K-independent Percolation On Infinite Trees After The Works Of Bollobas And Ballister

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Contents

List of Figures

List of Tables

1 Introduction

The study of percolation initially arose in the natural sciences, namely physics, chemistry and materials science. In physics percolation refers to a physical process that describes a transition of a system from one state to another, with a special focus on the appearance of an infinite size cluster corresponding to significant changes in the physical properties of the system. In chemistry and materials science percolation is understood as the movement and filtering of fluids through porous materials.

Mathematical percolation theory is the study of the structure and behaviour of random subgraphs generated by a probability measure on a given starting graph G. This master thesis has the goal to study certain classes of bond percolation on infinite trees, as their structure allows a more concise classification, and the question of the existence of an infinite cluster after percolation (the "physical view" of percolation).

The thesis starts with a characterization of infinite, locally finite and leafless trees in section 2.2. Based on the concepts of flows and cutsets in section 2.3 the branching number of a tree is introduced as a characteristic of its structure. It will be shown that for certain classes of percolations the branching number is the determining factor of their critical values.

A shot at probability theory is taken in section 3. Among the techniques the coupling of different Bernoulli random variables from section 3.2, later needed for independent bond percolation, and the crucial $0-1$ law from Kolmogorov in section 3.3 are indispensable to the later proofs.

Percolation is introduced in section 4, first in a totally general fashion as a probabiliy measure on the set of subgraphs of a given graph G and then restricting ourselves to the class of k-independent bond percolations on infinite, locally finite and leafless trees. After showing the connection between Kolmogorov's $0-1$ law and the question of the existence of an infinite cluster for a percolation in theorem 70 we introduce the tools needed to investigate percolation on trees: the switch between rooted and unrooted percolation (theorem 78), first and second moment methods (section 4.4) and random paths with exponential intersection tails in section 4.5 (see [9] and [11] for background on them).

Section 5 is an uptake of independent bond percolation, mostly following the treatment in [9]. It is self-contained, using only the methods from section 4 and not using explicitely the connection between network theory, homesick random walks on trees and percolation as in [9]. Its culmination is the determination of the critical value for independent percolation in terms of the branching number (see theorem 102 on page 34)

1-independent percolation on trees after the works of Bollobas and Balister [2] is treated in section 6: phase transition occurs here not at one value, but within a certain range (see figure 6 on page 64). To obtain the results we resuse the results from independent percolation and moment methods together with different Bernoulli models (see section 6.7).

This master thesis ends with some smaller results which can be generalized to k-independent percolation (section 7) and after a summary of the results in section 8.1 there is an outlook on open questions in section 8.2.

2 Graphs and trees

In this section I treat all the definitions and properties of graphs and claims, especially about infinite, locally finite, leafless trees, which can be derived from their combinatorial properties, needed in this work.

2.1 Basics on graphs

In this section one can find all the notation and definitions which are valid for general graphs, whereas those applicable only to trees are to be found in section 2.2.

Definition 1 *A* **graph** *is a pair $G = (V, E)$, where V is the countable set of* **vertices** *(also called* **nodes** *or* **sites***) and E is a symmetric subset of $V \times V$, called the* **edge** *set (also called* **bonds***). We say that G is* **finite** *iff V is finite, else G is called* **infinite***.*

For the remainder of this subsection let $G = (V, E)$ be a graph.

Definition 2 *Two nodes $x, y \in V$ are considered to be* **adjacent** *iff $(x, y) \in E$ and are called the* **endpoints** *of the edge $e = (x, y)$. Similarly, two edges $e, f \in E$ are called* **incident** *iff they have at least one common endpoint. Note that incidence and adjacency are transitive relations.*

Definition 3 *Take a second graph $H = (V', E')$. We say that G is a* **subgraph** *of H and equivalently H a* **supergraph** *of G iff there exists an injection $f : V \to V'$ which preserves adjacency and incidence, called an* **embedding** *of G into H. G and H are called* **isomorph** *iff they are embeddable into each other.*

Definition 4 *G is called a* **directed graph** *if we designate for each edge $e \in E$ an orientation (therefore E being any subset of $V \times V$).*

Definition 5 *We call the graph H constructed from G by creating a node for each edge in E and an edge for each couple of incident edges in E the* **line graph** *of G.*

Definition 6 *We call a* **sequence of subgraphs** $\{G_n = (V_n, E_n)\}_{n\in\mathbb{N}}$ **exhaustive** *iff it is ordered for set inclusion and $\bigcup_{n\in\mathbb{N}} G_n = G$.*

Definition 7 *For a node $x \in V$, we denote by $deg(x)$ the number of nodes adjacent to x, called the* **degree of** *x. We call G* **locally finite** *iff $\forall\, x \in E : deg(x) < \infty$ and* **uniformly bounded** *iff $\exists\, d \in \mathbb{N} : \forall\, x \in E : deg(x) \leq d$.*

Definition 8 *We call a* **path** *a sequence of edges in which each edge is incident to the next and previous ones in the sequence. Its* **length** *is defined to be the number of edges it traverses and subsequently it is called either* **finite** *or* **infinite***. For two nodes $x, y \in V$ we write $x \leftrightarrow y$ if there exists a path with endnodes x and y (and vice-versa, therefore being an equivalence relation). A* **cycle** *is a path of length greater than 1 having the same start- and endnode. We call a path* **self-avoiding** *if it doesn't visit any node or edge twice. It has* **unit speed** *if it contains a shortest path between all pairs of edges it contains and is cycleless, implying that it is self-avoiding. Finally a* **ray** *is an infinite path with unit speed.*

Definition 9 *The equivalence classes of V under connectedness are called* **connected components** *or* **clusters** *of G. We call G* **connected** *if it consists of only one connected component.*

Definition 10 *For $v, w \in V$ the* **node distance** *$|v - w|$ is the length of the shortest path from v to w. For two arcs $e, f \in E$ the* **arc distance** *it is the minimum of the shortest distances between any of their endnodes plus 1.*

It's plain to see that these two distances are metrics taking values only in $\mathbb{N}$. The following definition is taken from page 78 of [1].

Definition 11 *We call a subset Q of the set of all graphs which is closed under graph isomorphism a* **property of graphs***. A property Q is* **monotone increasing** *if whenever G has Q and H is a supergraph of G obtained by adding edges to G, then H has Q too. A* **monotone decreasing** *property is defined in a similar manner by the removal of edges.*

An example of monotone increasing properties is connectivity of the graph and for decreasing properties it would be cyclelessness of the graph.

2.2 Basics on trees

Definition 12 *A* **tree** *$T = (V, E)$ is a cycleless, connected graph. A graph whose connected components are all trees is called a* **forest***.*

Consequently there is always an unique shortest path between any two nodes of T and T is always a forest consisting of itself.

Definition 13 *If v has degree 1 then we call it a* **leaf***. A tree T is said to be* **leafless** *iff none of its nodes is a leaf, implying in particular that T is infinite.*

Definition 14 *T is called* r**-regular** *if* $\forall\, v \in V : deg(v) = r$.

Definition 15 *A* **rooted tree** *is a tree T with a distinguished node o, called the* **offspring** *or* **root**.

For the remainder of this section I assume T to be a rooted tree with root o, not excluding other conditions imposed on T by the following definitions. When picturing rooted trees we always place the root at the top with the rest of the tree spreading downwards (like in a family tree).

Definition 16 *For each node v we call the distance $|v - o| = l_v$ the* **level** *of v. The set of all nodes with level $n \in \mathbb{N}$ is written as $T_n = \{v \in V : l_v = n\}$.*

Definition 17 *In each rooted tree we have a* **canonical vertex-edge bijection** *between the nonroot vertices and edges of T. For all $v \in V \backslash \{o\}$ we denote by $e(v)$ the first edge lying on its shortest path towards the root and for $e \in E$ we denote by $v(e)$ the endpoint of e which is closer to the root.*

Definition 18 *For $v \in V \backslash \{o\}$, the* **parent** *of v, denoted by $p(v)$, is the unique neighbour of v with $l_{p(v)} + 1 = l_v$. We call* **ancestors** *of v the set $\{p(v), p(p(v)), \ldots, o\}$. For $v, w \in V$ we denote by $v \wedge w$ the* **common root** *of v and w, that means the common ancestor of v and w having the highest level.*

Definition 19 *For $v \in V$ we define $\{w \in V : p(w) = v\}$ as the set of* **children** *of v. The cardinality of this set is denoted by r_v and called the* **rank** *of v. It's plain to see that $r_v = 0 \Rightarrow deg(v) = 1 \Rightarrow v$ is a leaf. Likewise, we call the set containing all children of v, their children, ... and so on ad infinitum the* **offspring** *of v.*

Definition 20 *Through the child/parent relation the* **canonical partial order** *on T can be defined. For $v, w \in V$ we can write $v \leq w$ if w is a node on the unique path between v and the root o, including them, and $v < w$ if v is in the offspring of w or w is an ancestor of v.*

Definition 21 *For $v \in V$ we denote by T^v the* **subtree induced by** *v, formed by taking v and all its offspring and the arcs connecting them, effectivly creating a new rooted tree with root v.*

Definition 22 *For $n \in \mathbb{N}$ we denote by $T|_n$ the* **restricted subtree** *formed by taking all nodes v with $l_v \leq n$ and the arcs connecting them.*

Definition 23 *We denote by $\Upsilon(o)$ the* **set of all rays** *emanating from o.*

Definition 24 *If T is leafless we denote by the* **border** ∂T *of T the union of $\Upsilon(o)$ with the set of leaves of T. Note that if T is leafless, then $\partial T = \Upsilon(o)$ and if T is finite then ∂T is just the set of leaves of T.*

Definition 25 *If T is finite it is called a tree with $n \in \mathbb{N}$* **full levels** *if $\partial T = T_n$, meaning that all of its leaves are on the n^{th} level.*

Definition 26 *Let $N \geq 0$. Let T be infinite and rooted at o, then we call it*

$$\begin{aligned}
N\textbf{-periodic} \Leftrightarrow \forall\, v \in V: &\quad \exists\, f_v : T^v \to T^{f(v)} \text{ adjacency-preserving} \\
&\quad \text{bijective and } l_{f(v)} \leq N \\
N\textbf{-subperiodic} \Leftrightarrow \forall\, v \in V: &\quad \exists\, f_v : T^v \to T^{f(v)} \text{ adjacency-preserving} \\
&\quad \text{injective and } l_{f(v)} \leq N \\
N\textbf{-superperiodic} \Leftrightarrow \forall\, v \in V: &\quad \exists\, f_v : T \to T^{f(o)} \text{ adjacency-preserving} \\
&\quad \text{injective, } f(o) \in T^v \text{ and } l_{f(o)} - l_v \leq N
\end{aligned}$$

T is **(sub-/super-)periodic** *if it is so for some $N \in \mathbb{N}$.*

2.3 Flows and cutsets

The concepts of flow and cutset are applicable in the context of general graph, too. I just introduce the notation and relations between them as far as necessary in the context of trees. Let $T = (V, E)$ be a leafless tree rooted at $o \in V$. We start with

Definition 27 *A function $f : V \to \mathbb{R}$ is called a* **(sub-/super-)flow** *from o to infinity if*

$$\forall\, v \in V : f(v) = (\geq / \leq) \sum_{\{w : p(w) = v\}} f(w)$$

and is **nonnegative** *iff $\forall\, v \in V : f(v) \geq 0$. We call the value $St(f) = f(o)$ the* **strength** *of f and f the* **unit flow** *iff $f(o) = 1$. By the canonical vertex-edge bijection 17 we can change from a flow on V to a flow on E without any problems.*

Definition 28 *We call $\Pi \subseteq E$ a* **cutset separating o from infinity** *if the connected component of o in $T' = (V, E \backslash \Pi)$ is finite. We call Π a* **minimal cutset** *(for set inclusion) if $\forall\, e \in \Pi$: the connected component of o in $T' = (V, E \backslash (\Pi \backslash \{e\}))$ is infinite. We also denote the* **vertex-cutset** *$v(\Pi) = \{v(e) : e \in \Pi\}$ bijectively defined by Π.*

Proposition 29 *Assume T to be locally finite. Let Π be a cutset separating o from infinity, then there exists a minimal cutset $\Pi' \subseteq \Pi$ such that $|\Pi'| < \infty$.*

PROOF. Let C be the connected component of o in $T' = (V, E \backslash \Pi)$. Take $\Pi' = \{e \in \Pi : \exists\, v \in V(C) : e \text{ incident to } v\}$. Now finiteness of C in connection with the local finiteness of T implies the finiteness of Π'. □

Definition 30 *For T locally finite we denote by $\Pi(o)$ the* **set of all minimal cutsets separating o from infinity** *on T.*

So from now on we can talk safely about using only minimal, finite cutsets in the context of locally finite trees.

Definition 31 *Let Π be a cutset on T. We define the* **tree cut at Π**

$$T|_\Pi = (E|_\Pi = \{w \in V : \exists\, v \in v(\Pi) : w \leq v\}, \{e \in E : v(e) \in E|_\Pi\}),$$

consisting of the finite connected component of o in T determined by Π including Π and the vertex-cutset $v(\Pi)$.

Definition 32 *Let $\{\Pi_n\}_{n\in\mathbb{N}}$ be a sequence of cutsets on T. We call it an* **exhaustive sequence of cutsets**, *if the sequence of $\{T|_{\Pi_n}\}_{m\in\mathbb{N}}$ is an exhaustive sequence of subtrees of T.*

Next we explore the relationship between flows and cutsets.

Proposition 33 *Let Π be a minimal cutset and f be a (nonnegative) function on $v(\Pi)$, then*

$$\forall\, w \in V(T|_\Pi) : f(w) = \sum_{\{w \leq v \in v(\Pi)\}} f(v) \tag{1}$$

determines a flow on $T|_\Pi$ uniquely.

PROOF. This is proved by a recursion on w starting at $\partial T|_\Pi = v(\Pi)$.
$w \in v(\Pi)$: $f(w) = \sum_{\{w \leq v \in v(\Pi)\}} f(v)$ as Π is minimal.
$w \notin v(\Pi)$: Assume that (1) holds for all children w_i of w, then

$$f(w) = \sum_{i=1}^{r} f(w_i) = \sum_{i=1}^{r} \sum_{\{w_i \leq v \in v(\Pi)\}} f(v) = \sum_{\{w \leq v \in v(\Pi)\}} f(v)$$

using the disjointness of the subtrees starting in the w_i. □

Proposition 34 *Let f be a flow and Π a minimal cutset, then*

$$St(f) = \sum_{\{e \in \Pi\}} f(v(e)). \tag{2}$$

PROOF. Apply proposition 33 to o and Π. □

The following theorem is due to Ford and Fulkerson [5]:

Theorem 35 (Max-Flow Min-Cut) *Let c be a function from $E \to \mathbb{R}_+$ and T rooted at o, then*

$$\max_{\substack{f \text{ flow} \\ \forall e \in E:\, 0 \le f(e) \le c(e)}} St(f) = \inf_{\Pi \in \Pi(o)} \sum_{e \in \Pi} c(e). \tag{3}$$

Next we define a weighted quadratic form of a flow f.

Definition 36 *Let c be a function from $E \to \mathbb{R}_+$. Then the* **energy** *of f with respect to c is defined as*

$$\mathcal{E}(f)_c = \sum_{e \in E} \frac{f^2(v(e))}{c(e)}. \tag{4}$$

The following proposition, giving a sufficient condition for a flow having finite energy is taken from [9], proposition 2.22, page 51.

Proposition 37 (Finite flow energy) *Let $\{w_n\}_{n \in \mathbb{N}}$ be a positive sequence such that $\sum_{n \in \mathbb{N}} w_n = W < \infty$ and f a flow with $0 \le f(v) \le w_{l_v} c(e(v))$, then*

$$\mathcal{E}(f)_c \le St(f)W.$$

PROOF.

$$\begin{aligned}
&\mathcal{E}(f)_c \\
&= \sum_{e \in T} \frac{f(v(e))^2}{c(e)} \\
&= \sum_{n \in \mathbb{N}} \sum_{e \in E:\, l_{v(e)} = n} f(v(e)) \frac{f(v(e))}{c(e)} \\
&\le \sum_{n \in \mathbb{N}} \sum_{v \in V:\, l_v = n} f(v) w_n && \text{by the assumption} \\
&= \sum_{n \in \mathbb{N}} w_n \sum_{v \in V:\, l_v = n} f(v) \\
&= \sum_{n \in \mathbb{N}} w_n St(f) && \text{by proposition 34} \\
&= St(f)W
\end{aligned}$$

□

Lemma 38 (Nash-Williams) *Root T at o. Let $\{\Pi_n\}_{n\in A} \subset \Pi(o)$ be an exhaustive sequence of pairwise disjoint cutsets, where $A \subseteq \mathbb{N}$ is finite if T is finite. Let f be a unit flow from o and $c : E \to \mathbb{R}_+$. Then we have*

$$\mathcal{E}(f)_c \geq \sum_{n\in A} \left(\sum_{e\in\Pi_n} c(e) \right)^{-1}. \tag{5}$$

Proof. We start with showing that forall $\Pi \in \Pi(o)$ we have by Cauchy-Schwarz:

$$1 = \left(\sum_{e\in\Pi} f(e) \right)^2 = \left(\sum_{e\in\Pi} \frac{f(e)}{\sqrt{c(e)}} \sqrt{c(e)} \right)^2 \leq \sum_{e\in\Pi} \frac{f(e)^2}{c(e)} \sum_{e\in\Pi} c(e),$$

hence we arrive at

$$\mathcal{E}(f)_c = \sum_{e\in E} \frac{f(e)^2}{c(e)} \geq \sum_{n\in A} \sum_{e\in\Pi_n} \frac{f(e)^2}{c(e)} \geq \sum_{n\in A} \left(\sum_{e\in\Pi_n} c(e) \right)^{-1}.$$

□

2.4 Branching number of a tree

Next an important concept which will be used over and over again: λ-flows on rooted trees. Let $T = (V, E)$ be an infinite and locally finite tree, then

Definition 39 *A nonnegative flow f from $o \in V$ on T is called a* **lambda-flow** *for $\lambda > 0$ iff $\forall v \in E : f(v) \leq \lambda^{-l_v}$.*

Definition 40 *Let T be a tree rooted at o. Then its* **branching number** *with respect to o is defined by*

$$br(o, T) = \sup\{\lambda > 0 : \exists\, \lambda\text{-}flow\ on\ T\} \tag{6a}$$

$$= \sup\{\lambda > 0 : \inf_{\Pi\in\Pi(o)} \sum_{e\in\Pi} \lambda^{-l_{p(v(e))}} > 0\}. \tag{6b}$$

The equality between (6a) and (6b) is a result of theorem 35. The branching number of a rooted tree a is a kind of average rank of the nodes (see figure 1). This can be easily seen in the case of r-regular trees, for which the branching number is exactly $r - 1$ (upon rooting, all nodes except the root o have rank $r - 1$ and the root o has rank r).

Proposition 41 *$br(T) = br(o, T)$ is independent of the root o.*

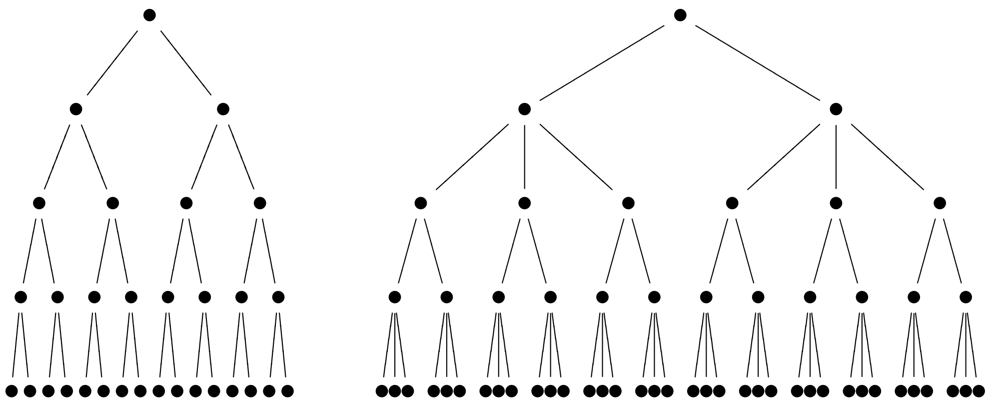

Figure 1: The rank 2 tree on the left has branching number 2, the tree on the right, with 2 and 3 children on alternating levels has branching number $\sqrt{6}$.

Proof. Any two different choices for the root $o, u \in V$ are only a finite distance of each other. We show that the branching number is the same for adjacent nodes, from which the result can be obtained for any distance by iterating over the path between o and u. For $\lambda < br(o,T)$ let f be a λ-flow from o. Let u be a child of o. Define a λ-flow g from u by

$$g(v) = \begin{cases} \frac{f(o)-f(u)}{\delta} & v = o \\ \frac{f(o)+f(u)}{\delta} & v = u \\ \frac{f(v)}{\delta} & \text{else,} \end{cases}$$

where $\delta = \max\{\lambda, \frac{\lambda}{f(o)-f(u)}\}$. Hence $br(o,T) \leq br(u,T)$, which results in $br(o,T) = br(u,T)$ by symmetry. □

I want to remark that the $br(T)$ of a tree equals the logarithm of the Hausdorff-Dimenstion of ∂T projected onto $[0,1]$ (see [6]).

Proposition 42 *The branching number is a monotone increasing property of infinite trees.*

Proof. Take a second infinite and locally finite tree $T' = (V', E')$ being a supertree of T. Select a root $o \in V$ and for $\lambda < br(T)$) take a λ-flow from o. f is clearly also a λ-flow on T', hence $br(T) \leq br(T')$. □

Proposition 43 $1 \leq br(T)$.

PROOF. Take a single ray – it has branching number 1. As it is contained in any infinite tree and the branching number is a monotone increasing property of trees the result follows. □

Proposition 44 *If T contains a leaf w, then it can be prunned without affecting the value of $br(T)$.*

PROOF. For $\lambda < br(T)$ take any λ-flow f from o. Now define a flow g from o by

$$g(v) = \begin{cases} f(v) - f(w) & w \leq v \\ f(v) & \text{else,} \end{cases}$$

effectively removing the amount $f(w)$ from all ancestors of w. g is again a λ-flow on $T' = (V \backslash \{w\}, E \backslash \{e(w)\})$, hence $br(T') \geq br(T)$. The reverse inequality is derived from the fact that T' is a subtree of T and by proposition 42. □

Lemma 45 *If $\exists\, o \in T$, such that if T is rooted at o, there exist at least two disjoint rays emanating from o, then prunning all leaves from T doesn't change the branching number of T.*

PROOF. Use proposition 44 to recursively prune all leaves from T. We have to demand two disjunct rays to exclude the degenerate cases with only one ray where recursive prunning would result in the empty tree. □

2.5 Growth of trees

Definition 46 *Let T be a locally finite, infinite tree rooted at o. We define the* **lower/upper growth rate** *of T by*

$$\underline{gr}(T) = \liminf_{n \to \infty} |T_n|^{\frac{1}{n}} \tag{7a}$$

$$\overline{gr}(T) = \limsup_{n \to \infty} |T_n|^{\frac{1}{n}} \tag{7b}$$

and call it the **growth rate** *$gr(T)$ of T iff $\underline{gr}(T) = \overline{gr}(T)$.*

Proposition 47 *We have*

$$br(T) \leq \underline{gr}(T) \leq \overline{gr}(T). \tag{8}$$

PROOF. Using 6b we show that

$$
\begin{aligned}
& br(T) \\
&= \sup\{\lambda > 0 : \inf_{\Pi\in\Pi(o)} \sum_{e\in\Pi} \lambda^{-l_{p(v(e))}} > 0\} \\
&\leq \sup\{\lambda > 0 : \inf_{n\in\mathbb{N}} \sum_{v\in T_n} \lambda^{-l_{p(v)}} > 0\} \\
&= \sup\{\lambda > 0 : \inf_{n\in\mathbb{N}} |T_n|\lambda^{-n} > 0\} \\
&= \sup\{\lambda > 0 : \exists\, c_\lambda > 0, N_\lambda \in \mathbb{N} : \forall\, n \geq N_\lambda : |T_n| \geq c_\lambda \lambda^n\} \\
&= \sup\{\lambda > 0 : \liminf_{n\to\infty} |T_n|^{\frac{1}{n}} \geq \lambda\} \\
&= \liminf_{n\to\infty} |T_n|^{\frac{1}{n}} \\
&= \underline{gr}(T).
\end{aligned}
$$

□

One can construct trees such that $br(T) < \underline{gr}(T)$ rather easily, for such an example see [9], page 4, example 1.2. The following two theorems are taken, too, from [9], pp 56–57, section 2.8, and are here stated without proof.

Theorem 48 *For every subperiodic tree T we have $br(T) = gr(T) < \infty$ and the existence of a $br(T)$-branching flow on T.*

Theorem 49 *Any superperiodic tree T with $\overline{gr}(T) < \infty$ satisfies $br(T) = gr(T)$.*

Corollary 50 *If $br(T) = gr(T)$ then*

$$\exists\, C, c > 0 : \forall\, n \in \mathbb{N} : c(br(T))^n \leq |T_n| \leq C(br(T))^n. \tag{9}$$

3 Probability

This section is dedicated to introduce all the notations and theorems, except the very foundational notations like σ-algebra, measure or random variable, related to general probability and (a bit of) measure theory which are needed throughout the rest of this work.

3.1 Reminders on probability and measure theory

The content of this section is paraphrased from [12]. Without really mentioning it in the future, we will always work over the probability space $(\Omega, \mathcal{A}, \mathbf{P})$. Random variables are always measurable with respect to $\mathcal{A}$. Furthermore expectation is denoted by $\mathbf{E}$ and indicator functions for any set A by $\mathbf{1}_A$. For two random variables X and Y we denote by $X \perp Y$ the fact that they are independent and by $X \not\perp Y$ the contrary.

Lemma 51 (Cauchy-Schwarz) *Let X, Y be $\mathbb{R}$-valued random variables such that $\mathbf{E}\, X^2 < \infty$ and $\mathbf{E}\, Y^2 < \infty$, then*

$$(\mathbf{E}\, XY)^2 \leq \mathbf{E}\, X^2 \mathbf{E}\, Y^2.$$

Theorem 52 (Fubini) *Let $(X, \mathcal{A}_X, \mu)$ and $(Y, \mathcal{A}_X, \nu)$ two spaces with their σ-algebras and measures on them. Let f be a $\mathcal{A}_X \otimes \mathcal{A}_Y$-measurable function such that $\int_{X \times Y} |f(x,y)| d\mu \times \nu(x,y) < \infty$, then*

$$\int_X \int_Y f(x,y) d\mu(x) d\nu(y) = \int_Y \int_X f(x,y) d\mu(y) d\nu(x).$$

Theorem 53 (Carathéodory) *Let X be a space, $\mathcal{A}$ be an algebra of its subsets and $\mathcal{A}_\sigma = \sigma(\mathcal{A})$ be the smallest σ-algebra containing $\mathcal{A}$. Let ν_0 be a σ-additive measure on $(X, \mathcal{A})$, then there exists a* **unique measure** *ν on $(X, \mathcal{A}_\sigma)$ which is an* **extension** *of ν_0, that means $\forall\, A \in \mathcal{A} : \nu_0(A) = \nu(A)$.*

Definition 54 *The sequence $\{X_n\}_{n \in \mathbb{N}}$ of random variables* **converges in probability** *to the random variable X, written as $X_n \xrightarrow[n \to \infty]{\mathbf{P}} X$, iff*

$$\forall\, \varepsilon > 0 : \mathbf{P}\left(|X_n - X| > \varepsilon\right) \xrightarrow[n \to \infty]{} 0.$$

3.2 Coupling and ordering Bernoulli random variables

The content of this section is taken from [7].

Definition 55 *Take two* $\mathbb{R}$*-valued random variables* X *and* Y*. We say that* X *is* **stochastically less than** Y *(written as* $X \leq_{st} Y$*) iff*

$$\forall c \in \mathbb{R} : \mathbf{P}(X > c) \leq \mathbf{P}(Y > c).$$

Next we describe how to couple a random variable $X\sim$ Bernoulli(p) to a random variable $Y\sim$ Unif($[0,1]$) with uniform distribution over the unit interval: simply couple them by setting $X = \mathbf{1}_{\{Y>p\}}$. We show a little proposition demonstrating the relationship between different Bernoulli(p) random variables coupled to the same Unif($[0,1]$).

Proposition 56 *Let* $X\sim$ *Bernoulli(*p*),* $X'\sim$ *Bernoulli(*p'*) both be coupled to* $Y\sim$ *Unif(*$[0,1]$*), then we have*

$$p < p' \Leftrightarrow X <_{st} X' \text{ and } p \leq p' \Leftrightarrow X \leq_{st} X'. \tag{10}$$

Proof. It's plain to see from the coupling described above. □

Proposition 57 *Let* $X_{p_n}\sim$ *Bernoulli(*p_n*),* $X\sim$ *Bernoulli(*p*) be both coupled to* $Y\sim$ *Unif(*$[0,1]$*), then*

$$p_n \xrightarrow[n\to\infty]{} p \Rightarrow X_{p_n} \xrightarrow[n\to\infty]{\mathbf{P}} X_p. \tag{11}$$

Proof. Let $\varepsilon > 0$ and suppose $p_n < p$, then we have

$$\mathbf{P}(|X_{p_n} - X_p| > \varepsilon) = \mathbf{P}(p - \varepsilon \geq p_n < Y \leq p) \leq |p_n - p| \xrightarrow[n\to\infty]{} 0.$$

□

3.3 Kolmogorov's $0-1$ law

This section is taken from [12], chapter IV.1, pp 379–381.

Definition 58 *Take* $\{X_i\}_{i\in\mathbb{N}}$ *be a sequence of independent random variables. Let* $\mathcal{A}_1^j = \sigma(X_1, \ldots, X_j)$ *and* $\mathcal{A}_j^\infty = \sigma(X_j, X_{j+1}, \ldots) = \sigma(\bigcup_{i=j}^\infty \mathcal{A}_k^i)$ *then the* **tail** σ**-algebra** *of* $\{X_i\}_{i\in\mathbb{N}}$ *is defined as*

$$\mathcal{A}^\infty = \bigcap_{i=1}^\infty \mathcal{A}_i^\infty.$$

We call A *a* **tail event** *iff* $A \in \mathcal{A}^\infty$*, that means an event which is independent of any finite subset of* $\{X_i\}_{i\in\mathbb{N}}$*.*

Theorem 59 (Kolmogorov's $0-1$ law) *Take the setting of definition 58. Let $A \in \mathcal{A}^{\infty}$. Then*

$$\mathbf{P}(A) \in \{0,1\}.$$

We omit the proof as the next theorem is an extension of Kolmogorov's $0-1$ law 59 for locally dependent variables which subsumes the former theorem.

Definition 60 *We call a family of random variables $\{X_i\}_{i\in\mathbb{N}}$* **locally dependent** *iff $\forall\, i \in \mathbb{N} : |\{j \in \mathbb{N} : X_i \not\perp X_i\}| < \infty$.*

Theorem 61 (Extension of Kolmogorov's $0-1$ law) *For any locally dependent family random variables and any event A in their tail σ-algebra we have*

$$\mathbf{P}(A) \in \{0,1\}.$$

PROOF. First we adapt our σ-algebras accordingly. Define by $D_i = \{X_j : X_i \not\perp X_j\}$ the finite set of dependencies of X_i. Then set $\mathcal{A}_1^j = \sigma(\bigcup_{i=1}^{j} D_i)$, $\mathcal{A}_j^{\infty} = \sigma(\bigcup_{i=j}^{\infty} D_i)$ and the tail σ-algebra $\mathcal{A}^{\infty} = \bigcap_{i=1}^{\infty} \mathcal{A}_i^{\infty}$.

Now take an event $A \in \mathcal{A}^{\infty}$. Define $\forall\, i \in \mathbb{N} : Z_i = \mathbf{E}[\mathbf{1}_{\{A\}}|\mathcal{A}_1^i]$, which is a martingale, as

$$\mathbf{E}[Z_{i+1}|\mathcal{A}_1^i] = \mathbf{E}[\mathbf{E}[\mathbf{1}_{\{A\}}|\mathcal{A}_1^{i+1}]|\mathcal{A}_1^i] = \mathbf{E}[\mathbf{E}[\mathbf{1}_{\{A\}}|\mathcal{A}_1^i]|\mathcal{A}_1^{i+1}] = \mathbf{E}[\mathbf{1}_{\{A\}}|\mathcal{A}_1^i] = Z_i.$$

Furthermore for each i, there exists a

$$j(i) = \min\{j : \forall\, X\ \mathcal{A}_1^i\text{-measurable}\ \forall\, Y\ \mathcal{A}_j^{\infty}\text{-measurable} : X \perp Y\}$$

such that $\mathcal{A}_{j(i)}^{\infty}$ independent of $\mathcal{A}_1^i$, therefore A is independent of $\mathcal{A}_1^i$. Hence $\forall\, i \in \mathbb{N} : Z_i = \mathbf{P}(A)$ is constant. As $A \in \mathcal{A}^{\infty} \subseteq \bigcup_{i=1}^{\infty} \mathcal{A}_1^{\infty}$ Z_i converges in L_1 and almost surely to $Z_{\infty} = \mathbf{1}_{\{A\}}$, thus

$$\forall\, i \in \mathbb{N}, \forall\, X \mathcal{A}_1^i\text{-measurable} : \mathbf{E}[Z_{\infty} X] = \mathbf{E}[\mathbf{1}_{\{A\}} X].$$

By taking $X = Z_i$ we get

$$\mathbf{P}(A)^2 = \mathbf{E}[\mathbf{1}_{\{A\}}\mathbf{P}(A)] = \mathbf{E}[Z_{\infty} Z_i] \xrightarrow[i\to\infty]{} \mathbf{E}[Z_{\infty}] = \mathbf{E}[\mathbf{1}_{\{A\}}] = \mathbf{P}(A).$$

The only solutions of the above equation are $\mathbf{P}(A) = 0$ or $\mathbf{P}(A) = 1$. □

3.4 K-independent random variables

The following definition is taken from [2], the rest is simple calculus.

Definition 62 *Let $\{X_i\}_{i\in\mathbb{N}}$ be a family of random variables and let $d : \mathbb{N}^2 \to \mathbb{R}_+$ be a* **distance function** *over the indices. We call the* **family k-independent** *iff $\forall\, A, B \subset \mathbb{N}$:*

$$d(A, B) = \min\{d(a, b) : a \in A, b \in B\} > k \quad \Rightarrow \quad \{X_a\}_{a\in A} \perp \{X_b\}_{b\in B}.$$

I want to point out that a 0-independent family of random variables is just the special case of an independent family of random variables.

Definition 63 *We denote for $p \in]0, 1[$ and $k, l \in \mathbb{N}$ by $\{X_i\}_{i=1}^{l}$ a* **linear k-independent** *family of Bernoulli(p) random variables, where the distance function is $d(i, j) = |i - j|$ (think of the variables indexed by $\{1, \ldots, n\}$).*

In the following we will use such random variables and derive an upper bound on joint probabilities of them.

Proposition 64 *Let $\{X_i\}_{i=1}^{l}$ be as above, then we have*

$$\mathbf{P}\left(X_1 = \ldots = X_l = 1\right) \leq p^{\left\lceil \frac{l}{k+1} \right\rceil}. \tag{12}$$

PROOF.

$$\begin{aligned}
&\mathbf{P}\left(X_1 = \ldots = X_l = 1\right) \\
&\leq \mathbf{P}\left(X_1 = X_{k+2} = X_{2k+3} = \ldots = X_{1+(k+1)\left\lfloor \frac{l}{k+1} \right\rfloor} = 1\right) \\
&= p^{1+\left\lfloor \frac{l}{k+1} \right\rfloor} = p^{\left\lceil \frac{l}{k+1} \right\rceil}.
\end{aligned}$$

□

4 Percolation

Take a graph $G = (V, E)$. Percolation is the study of the comportment of the clusters of random subgraphs $G(\omega)$ of G. We are especially interested in the macroscopic behaviour of $G(\omega)$: what is the probability of it containing an infinite cluster? In this context we say that we "percolate" iff we have a cluster of infinite size in $G(\omega)$.

Closely connected to this is the notion of **critical probabilities** – bounds on almost sure existence, respectively non-existence, of such an infinite cluster. This leads to other questions such as the behaviour of the percolation at the critical probability, called the **phase transition** or the properties of the infinite cluster after percolation has taken place.

4.1 Definition of percolation

Definition 65 *Given a countable set A with a transitive relation over A (which can be represented by a directed graph) we can characterize* **percolation on** A *by either*

1. *a family of Bernoulli random variables $\{X_a\}_{a\in A}$ with*

$$\forall\, a \in A : \mathbf{P}\left(X_a = 1 = \textit{open}\right) + \mathbf{P}\left(X_a = 0 = \textit{closed}\right) = 1$$

2. *or a probability measure on $\{0,1\}^{|A|}$, the power set of A.*

On setting $X_a = \mathbf{1}_{\{a(\omega)=1\}}$ one easily sees the equivalence of the two characterizations. For $a \in A$ we denote by

$$R_a = \{b \in A : b \text{ related to } a \text{ via a finite sequence of relations}\}, \tag{13}$$

quietly assuming that $\exists\, a \in A : R_a = A$. We are now interested in the event

$$\{\omega : \exists\, a \in A : \exists\, F \subseteq R_a(\omega) : |F| = \infty\} = \{\exists \text{ cluster of } \infty \text{ size in } A\}. \tag{14}$$

We say that a percolation **percolates** if we are in the above event. Note that if the relation on A is symmetric, then A can be represented as a graph, otherwise as a directed graph. Percolation then can be seen as a probability measure over the set of (directed) subgraphs of the initial (directed) graph representation of A. Finally, an important class of percolations worth mentioning is

Definition 66 *We call a percolation $\mathcal{P}$ on A a* **percolation with parameter** p *iff*

$$\forall\, a \in A : \mathbf{P}\left(X_a = 1\right) = p.$$

4.2 Percolation on graphs

We now take a graph $G = (V, E)$. There are many ways to classify percolation measures on G, following are the ones needed throughout the rest of this paper. A first classification is based on which the set A is used:

Definition 67 *Let* $\mathcal{P}$ *be a* **percolation on** G*. Then we call* $\mathcal{P}$

$$\text{a } \textbf{node/site percolation} \Leftrightarrow V = A \qquad \textit{(via node adjacency)} \tag{15a}$$
$$\text{an } \textbf{edge/bond percolation} \Leftrightarrow E = A \qquad \textit{(via edge incidence)} \tag{15b}$$

Here we use the fact that incidence and adjacency (see 2) are equivalence relations, implying that they are symmetric and transitive. I want to remark that any bond percolation on G can be expressed as a site percolation on the line graph (see 5) of G (see [8], section 1.6, page 24). In the following we only talk about bond percolations, but the definitions stay the same for site percolations.

Definition 68 *Let* $\mathcal{P}$ *be a percolation on* E*. Then we call* $\mathcal{P}$ **k-independent** *if the distance function is the canonical arc distance on* E*. Furthermore we distinguish between* **real k-independent percolations***, which are percolations being* k *but not* $k-1$*-independent and* **full k-independent percolations***, which are percolations which satisfy the additional property:* $\forall\, e, f \in E : |e - f| \leq k \Rightarrow X_e \not\perp X_f$.

Note that 0-independent percolation is synonymous to independent percolation.

Definition 69 *In the following, for a graph* $G = (V, E)$ *and* $p \in [0, 1]$ *we will use* $\mathcal{C}_p^k(E)$ *to denote the* **class of all k-independent bond percolations with parameter p on** E *and similarly for site percolation* $\mathcal{C}_p^k(V)$*. If* $k = 0$ *then* k *might be omitted. Similarly, percolations in the above classes are written like* $\mathcal{P}_p^k(E)$ *as an example for a* **k-independent bond percolation with parameter** p*.*

Next we state a theorem which is used as an important building block for later proofs:

Theorem 70 ($0-1$ law for k-independent percolation) *On any locally finite graph* $G = (V, E)$ *and for any percolation* $\mathcal{P} \in \mathcal{C}_p^k(V)$ *or* $\mathcal{P} \in \mathcal{C}_p^k(E)$ *we have*

$$\mathbf{P}\,(\mathcal{P} \textit{ percolates}) \in \{0, 1\}.$$

PROOF. Let $B = \{\omega : \mathcal{P} \text{ percolates}\}$ be the event defined in (14). Clearly it is in the tail σ-algebra of the random variables $\{X_a\}_{a\in A}$, as it doesn't depend on the state of finitely many elements X_a (where A may be V or E). As G is locally finite and the $\{X_a\}_{a\in A}$ are k-independent they are also locally dependent (see definition 60), therefore one can conclude by the extension of Kolmogorov's $0-1$ law 61. □

4.3 Percolation on trees

For discussing bond percolation on trees we assume $T = (V, E)$ to be an infinite and locally finite tree. Furthermore let $\mathcal{P} \in \mathcal{C}_p^k(E)$, for finite k. The first thing that we show is that we can assume without loss of generality leaflessness of the trees.

Proposition 71 *If T contains a node w which is a leaf, it can be prunned without changing the percolation behaviour of $\mathcal{P}$ on T.*

PROOF. Root T at $o \neq w$ and let $T' = (V\backslash\{w\}, E\backslash\{e(w)\})$ be the tree of which w has been prunned. As by theorem 61 the event that $\mathcal{P}$ percolates on T is is independent of the finite set $\{X_v : |v-w| \leq k\}$, hence it is the same event on T'. Applying theorem 70 yields the desired result. □

Proposition 72 *If T can be rooted at a node o such that there are at least two distinct rays from o, then prunning all leaves from T doesn't change the percolation behaviour of T.*

PROOF. Use proposition 71 to recursively prune all leaves from T. The condition of having at least two rays guarantees that we don't prune the whole tree. □

So from now on we can safely assume that T is leafless. The next step is exploring the connection between percolation on the rooted tree and the tree itself. Starting with a definition:

Definition 73 *Let T be rooted at o. Consider the percolation $\mathcal{P} \in \mathcal{C}_p^k(E)$. Let $n \in \mathbb{N}$ and $w \in T|_n$, then we call the event*

$$O_w^n = \{\omega : \exists\, v \in T_{n-l_w}^w(\omega) : w \leftrightarrow v\} = \{w \leftrightarrow n\}$$

"w has a **downpath to level** *n", that means only using edges in its induced subtree and paths to nodes in T_n. Similarly the event*

$$O_w = \{w \leftrightarrow \infty\} = \{\omega : \exists \text{ ray from } w \text{ in } T^w(\omega)\}$$

is simply called "w has a **downray***".*

Notation 74 *Based on definition 73 we introduce the following notation:*

$$\begin{array}{lll} \mathbf{P}(O_w^n) = \xi_w^n & F_w^n = (O_w^n)^c & \mathbf{P}(F_w^n) = \eta_w^n \\ \mathbf{P}(O_w) = \xi_w & F_w = O_w^c & \mathbf{P}(F_w) = \eta_w \end{array}$$

and furthermore for $w \neq o$: $E_w = \{\omega : e(w) \text{ closed in } T(\omega)\}$ the event that the edge $e(w)$ is closed.

Next we see how to switch between percolation on the rooted tree and reaching for all $\Pi \in \Pi(o)$ $\partial T|_\Pi$ from the root o.

Proposition 75 *Let T be rooted at o. Then for any percolation $\mathcal{P} \in \mathcal{C}_p^k(E)$ and any $w \in V$ we have*

$$\bigcap_{n \geq l_w} O_w^n = O_w \tag{16a}$$

$$O_w^n \xrightarrow[n\to\infty]{\mathbf{P}} O_w \tag{16b}$$

Proof. We start by proving (16a):
$\supseteq$: If there is a downray from w, then its first n edges form a downpath of length n to level n (as a ray doesn't use edges twice it can only descend down the tree). As this can be done $\forall\, n \in \mathbb{N}$ the claim follows.
$\subseteq$: Now take $\omega \in \bigcap_{n\geq l_w} O_w^n \Rightarrow \forall\, n \geq l_w : \exists\, P_n$, which is a path from w to level n. Now starting with w and descending recursively we start the following procedure: as T is locally finite, there are only finitely many children of w, hence there exists a child w_i of w through which infinitely many of these paths $\{P_n\}_{n\geq l_w}$ pass. Recurse on that child w_i. We get an infinite sequence of nodes, respectively open edges, which form a ray.
To prove (16b) we only have to use (16a) and observe that $\{O_w^n\}_{n\geq l_w}$ is a decreasing sequence of events. □

Corollary 76 *Let T be rooted at o, $\mathcal{P} \in \mathcal{C}_p^k(E)$ be any percolation, $\{\Pi_m\}_{m\in\mathbb{N}}$ any sequence of exhausting cutsets of T and $w \in V$, then we have:*

$$\bigcap_{m \geq m(w)} O_w^{\Pi_m} = O_w \tag{17a}$$

$$O_w^{\Pi_m} \xrightarrow[m\to\infty]{\mathbf{P}} O_w \tag{17b}$$

Proof. $O_w^{\Pi_m}$ is of course the event of having a downpath down to the cutset Π_m. Now set $m(w) = \min_{m\in\mathbb{N}} \{T|_{l_w} \subseteq T|_{\Pi_m}\}$, which exists as $\{\Pi_m\}_{m\in\mathbb{N}}$ is exhaustive. For $m \geq m(w)$ take $n(m) = \max_{n\in\mathbb{N}} \{T|_n \subseteq T|_{\Pi_m}\}$. The sequence

$\{n(m)\}_{m\in\mathbb{N}}$ is monotone increasing as $\{\Pi_m\}_{m\in\mathbb{N}}$ is exhaustive. Furthermore $O_w \subseteq O_w^{\Pi_m}$ as all downrays from w have to pass by Π_m, hence

$$O_w \subseteq \bigcap_{m\geq m(w)} O_w^{\Pi_m} \subseteq \bigcap_{m\geq m(w)} O_w^{n(m)} = \bigcap_{n\geq l_w} O_w^n = O_w,$$

which proves (17a). (17b) follows from the fact that $\{O_w^{\Pi_m}\}_{m\in\mathbb{N}}$ is a decreasing sequence of events. □

Theorem 77 *We have*

$$O_o = \bigcap_{\Pi\in\Pi(o)} O_o^{\Pi}, \tag{18a}$$

and

$$\mathbf{P}(o\leftrightarrow\infty) = \xi_o = \inf_{\Pi\in\Pi(o)} \xi_o^{\Pi} = \inf_{\Pi\in\Pi(o)} \mathbf{P}(o\leftrightarrow\Pi) \tag{18b}$$

PROOF. The one direction is clear as $O_o \subseteq O_o^{\Pi}$ for any $\Pi\in\Pi(o)$. The other one follows from applying corollary 76 with any exhaustive sequence of cutsets $\{\Pi_m\}_{m\in\mathbb{N}}$. □

And finally the link between percolation on rooted trees to percolation on the unrooted tree itself.

Theorem 78 *Let $\mathcal{P}\in\mathcal{C}_p^k(E)$, for finite k. Then*

$$(\exists\, v\in V : \xi_v > 0) \Leftrightarrow \mathbf{P}(\mathcal{P} \textit{ percolates on } T) = 1, \tag{19a}$$
$$(\forall\, v\in V : \xi_v = 0) \Leftrightarrow \mathbf{P}(\mathcal{P} \textit{ percolates on } T) = 0. \tag{19b}$$

PROOF. As the two statements are equivalent, we will just prove (19a).
$\Rightarrow$: Take $v\in V$ such that $\mathbf{P}(O_v) > 0$. The downray from v forms an infinite cluster, hence $\mathcal{P}$ percolates on T with nonzero probability and by theorem 70 with probability 1. In set theoretic notation this is:

$$\forall\, v\in V : O_v \subseteq O \Leftrightarrow \bigcup_{v\in V} O_v \subseteq O \Leftrightarrow O^c \subseteq \bigcap_{v\in V} O_v^c$$

$\Leftarrow$: Root T at $o\in V$. We now regard the random variable m_o, designating the node of the infinite cluster having the highest level (being closest to the root o). We now have

$$\mathbf{P}(\mathcal{P} \text{ percolates on } T) = 1 \Rightarrow \mathbf{P}(m_o\in V) = 1 \Rightarrow \exists\, w\in V : \mathbf{P}(m_o = w) > 0$$

which by the preceding equation and on setting $o = w$ results in

$$\xi_w = \mathbf{P}\left(O_w\right) = \sum_{v \in V} \mathbf{P}\left(O_w, m_w = v\right) \geq \mathbf{P}\left(O_w, m_w = w\right) = \mathbf{P}\left(m_w = w\right) > 0.$$

□

As percolation creates a random subgraph of a general graph it creates a random forest from a tree (see figure 2). To study the macroscopic behaviour of it, especially the existence of an infinite tree within the random forest, we define two values which are of principal interest for its characterization:

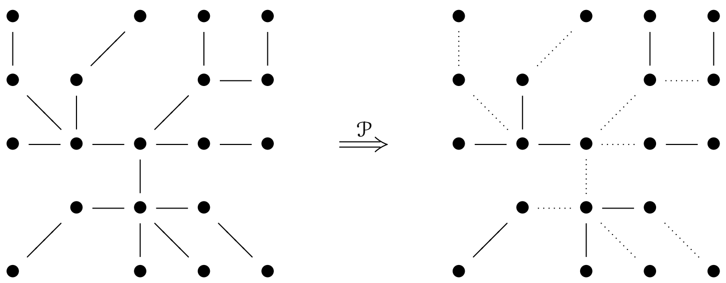

Figure 2: Random forest after percolation. Edges marked "…" in the right graph have been deleted by the percolation $\mathcal{P}$.

Definition 79 *For $k \in \mathbb{N}$ and T a leafless, locally finite tree with finite branching number define the* **critical values** *of T as*

$$p^k_{\max}(T) = \inf_{p \in [0,1]} \{\forall\, \mathcal{P}^k_p(E) \in \mathcal{C}^k_p(E) : \mathcal{P}^k_p(E) \text{ percolates}\} \tag{20}$$

$$p^k_{\min}(T) = \inf_{p \in [0,1]} \{\exists\, \mathcal{P}^k_p(E) \in \mathcal{C}^k_p(E) : \mathcal{P}^k_p(E) \text{ percolates}\} \tag{21}$$

Lemma 80 *For any tree T we have*

$$\ldots \leq p^2_{\min}(T) \leq p^1_{\min}(T) \leq p^0_{\min}(T) \leq p^0_{\max}(T) \leq p^1_{\max}(T) \leq p^2_{\max}(T) \leq \ldots.$$

Proof. $p^0_{\min}(T) \leq p^0_{\max}(T)$ follows straight from definition 79. For the other inequalities note that we have

$$\mathcal{C}^2_p(E) \subseteq \mathcal{C}^1_p(E) \subseteq \ldots \subseteq \mathcal{C}^k_p(E) \subseteq \ldots$$

which implies $\forall\, k \in \mathbb{N} : \quad p^{k+1}_{\min}(T) \leq p^k_{\min}(T) \quad \wedge \quad p^k_{\max}(T) \leq p^{k+1}_{\max}(T)$. □

4.4 Moment methods

In this section I present the 1^{st} and 2^{nd} moment methods as shown in [9], sections 4.2/4.3, pp 101. The 2^{nd} moment method uses weighted energies of probability measures on cutsets to give upper bounds for ξ_o^Π and therefore ξ_o, while the 1^{st} moment method yields lower bounds.
Let T be a locally finite, leafless tree rooted at o with finite branching number. Let $\mathcal{P} \in \mathcal{C}_p^k(E)$, for finite k. Denote the **connected component of** o **under percolation** by $C(o)$. For an edge e write $\mathbf{P}_{C(o)}(e) = \mathbf{P}\left(e \in C(o)\right)$.

Proposition 81 (1^{st} moment method) *Under these settings we have*

$$\xi_o = \mathbf{P}\left(o \leftrightarrow \infty\right) \leq \inf_{\Pi \in \Pi(o)} \sum_{e \in \Pi} \mathbf{P}_{C(o)}(e) \tag{22}$$

Proof. For any $\Pi \in \Pi(o)$ we have

$$O_o^\Pi = \{o \leftrightarrow \Pi\} = \{\exists\, e \in \Pi : e \in C(o)\} \subseteq \bigcup_{e \in \Pi} \{e \in C(o)\}$$

which, using (18a), results in

$$O_o = \{o \leftrightarrow \infty\} = \bigcap_{\Pi \in \Pi(o)} O_o^\Pi \subseteq \bigcap_{\Pi \in \Pi(o)} \bigcup_{e \in \Pi} \{e \in C(o)\}.$$

□

Definition 82 *For all* $\Pi \in \Pi(o)$ *and* $\mu \in \mathcal{M}_1(\Pi)$ *define the* **percolation function** *of* μ *by*

$$\chi(\mu) = \sum_{e \in \Pi} \mu(e) \frac{\mathbf{1}_{\{e \in C(o)\}}}{\mathbf{P}_{C(o)}(e)}. \tag{23}$$

Proposition 83 *The percolation function has the following properties:*

$$\mathbf{E}\,\chi(\mu) = 1 \tag{24a}$$

$$\mathbf{E}\,\chi(\mu)^2 = \sum_{e,f \in \Pi} \mu(e)\mu(f) \frac{\mathbf{P}_{C(o)}(e,f)}{\mathbf{P}_{C(o)}(e)\mathbf{P}_{C(o)}(f)} \tag{24b}$$

Proof. First we prove (24a):

$$\mathbf{E}\,\chi(\mu) = \mathbf{E} \sum_{e \in \Pi} \mu(e) \frac{\mathbf{1}_{\{e \in C(o)\}}}{\mathbf{P}_{C(o)}(e)} = \sum_{e \in \Pi} \mu(e) \frac{\mathbf{P}_{C(o)}(e)}{\mathbf{P}_{C(o)}(e)} = \sum_{e \in \Pi} \mu(e) = 1.$$

For (24b) we show that

$$\begin{aligned}
&\mathbf{E}\,\chi(\mu)^2\\
&= \mathbf{E}\left[\sum_{e\in\Pi}\mu(e)\frac{\mathbf{1}_{\{e\in C(o)\}}}{\mathbf{P}_{C(o)}(e)}\right]^2\\
&= \sum_{e,f\in\Pi}\mu(e)\mu(f)\mathbf{E}\,\frac{\mathbf{1}_{\{e\in C(o)\}}\mathbf{1}_{\{f\in C(o)\}}}{\mathbf{P}_{C(o)}(e)\mathbf{P}_{C(o)}(f)}\\
&= \sum_{e,f\in\Pi}\mu(e)\mu(f)\frac{\mathbf{P}_{C(o)}(e,f)}{\mathbf{P}_{C(o)}(e)\mathbf{P}_{C(o)}(f)}
\end{aligned}$$

□

Definition 84 *Define the* **percolation edge kernel** *of* $\mathcal{P}$ *at* Π *by*

$$\kappa_\Pi : \qquad \Pi^2 \to \mathbb{R}_+ \qquad (e,f)\mapsto \kappa_\Pi(e,f) = \frac{\mathbf{P}_{C(o)}(e,f)}{\mathbf{P}_{C(o)}(e)\mathbf{P}_{C(o)}(f)}. \tag{25}$$

The percolation edge kernel factors the common part out of (24b). Note that it is symmetric in its arguments, which allows us to define:

Definition 85 *We call the* **second moment** *of* $\chi(\mu)$ *the* **energy** *of* μ *because of its resemblance to a quadratic form. Its value is*

$$\mathcal{E}(\mu) = \sum_{e,f\in\Pi}\mu(e)\mu(f)\kappa_\Pi(e,f). \tag{26}$$

Lemma 86 (2^{nd} moment method) *Under these settings we have*

$$\xi_o = \mathbf{P}\left(o\leftrightarrow\infty\right) \geq \inf_{\Pi\in\Pi(o)}\ \sup_{\mu\in\mathcal{M}_1(\Pi)}\frac{1}{\mathcal{E}(\mu)}. \tag{27}$$

Proof. First we note that

$$\{\chi(\mu)>0\} = \{\exists\, e\in\Pi : e\in C(o)\} = \{o\leftrightarrow\Pi\},$$

which, with the help of Cauchy-Schwarz 51, we use to show that

$$\begin{aligned}
1 = 1^2 = \mathbf{E}^2\chi(\mu) &= \mathbf{E}^2[\chi(\mu)\mathbf{1}_{\{\chi(\mu)>0\}}] \leq \mathbf{E}\,\chi(\mu)^2\mathbf{E}\,[\mathbf{1}_{\{\chi(\mu)>0\}}]^2\\
&= \mathbf{E}\,\chi(\mu)^2\mathbf{P}\left(\chi(\mu)>0\right) \leq \mathcal{E}(\mu)\mathbf{P}\left(o\leftrightarrow\Pi\right),
\end{aligned}$$

hence $\mathbf{P}\left(o\leftrightarrow\Pi\right) \geq \frac{1}{\mathcal{E}(\mu)}$ from which, using (18b), we deduce that

$$\xi_o = \mathbf{P}\left(o\leftrightarrow\infty\right) = \inf_{\Pi\in\Pi(o)}\xi_o^\Pi \geq \inf_{\Pi\in\Pi(o)}\ \sup_{\mu\in\mathcal{M}_1(\Pi)}\frac{1}{\mathcal{E}(\mu)}.$$

□

4.5 Random paths and exponential intersection tails

Here we explore the connection between probability measures on $\Upsilon(o)$ and the 2^{nd} moment method. Note that in the context of trees rays are equivalent to paths having unit speed and also to self-avoiding paths. This section is based, with slight modifications, on [11], sections 10 and 11. Let T be a locally finite, leafless tree rooted at o with finite branching number. We first start with a useful bijection between flows and measures on rays.

Definition 87 *Let $\psi, \varphi \in \Upsilon(o)$ and $\Pi \in \Pi(o)$. Then we denote the* **ray stopped at** Π *by φ_Π, which is the part of φ lying in $T|_\Pi$, and similarly by ψ_N the* **ray stopped at level** *N, which are the first N edges of ψ starting from o. Furthermore we denote by $\psi \wedge \varphi$ the* **splitting node** *of ψ and φ, which is the node with the lowest level in $\psi \cap \varphi$.*

Proposition 88 *There is a* **bijection between** *flows with finite strength from o and $\mathcal{M}(\Upsilon(o))$, the set of finite measures on $\Upsilon(o)$, as well as between* **nonnegative unit flows** *and $\mathcal{M}_1(\Upsilon(o))$, the* **set of probability measures on** $\Upsilon(o)$.

Proof. We only show here the second part, as the first one is a trivial generalization of it. For f a nonnegative unit flow from o and $\nu \in \mathcal{M}_1(\Upsilon(o))$ associate the two with each other if

$$\forall\, e \in E : f(e) = \nu(\varphi : e \in \varphi).$$

Remark that ν is uniquely determined by its values over the cylinder sets $\{\varphi : e \in \varphi\}$, as its extension to a probability measure on $\Upsilon(o)$ is unique by Carathéodory 53. Clearly this association is a bijection which respects all the conditions like nonnegativity and $f(o) = 1 = \nu(\Upsilon(o))$. □

Proposition 89 *Let $\nu \in \mathcal{M}_1(\Upsilon(o))$, then on each $\Pi \in \Pi(o)$ it induces a probability measure $\nu|_\Pi \in \mathcal{M}_1(\Pi)$.*

Proof. For $\Pi \in \Pi(o)$ define $\nu|_\Pi \in \mathcal{M}_1(\Pi)$ by

$$\forall\, e \in \Pi : \nu|_\Pi(e) = \nu(\varphi : e \in \varphi), \tag{28}$$

which is the **projection** of ν on Π. □

Next we see, that if we have such an induced probability measure μ on Π, that we can rewrite $\mathcal{E}(\mu)$ in a different form.

Definition 90 *Define the* **percolation ray kernel** *of* $\mathcal{P}$ *at* Π *by*

$$K_\Pi : \qquad \Upsilon(o)^2 \to \mathbb{R}_+ \qquad (\varphi, \psi) \mapsto \frac{\mathbf{P}_{C(o)}(\varphi_\Pi \cup \psi_\Pi)}{\mathbf{P}_{C(o)}(\varphi_\Pi)\mathbf{P}_{C(o)}(\psi_\Pi)}. \tag{29}$$

Lemma 91 *For* $\nu \in \mathcal{M}_1(\Upsilon(o))$ *and* $\Pi \in \Pi(o)$ *we have*

$$\mathcal{E}(\nu|_\Pi) = \int_{\Upsilon(o)^2} K_\Pi(\varphi, \psi) d\nu \times \nu(\varphi, \psi). \tag{30}$$

Proof. We start by writing

$$\begin{aligned}
&\chi(\nu|_\Pi) \\
&= \sum_{e \in \Pi} \nu|_\Pi(e) \frac{\mathbf{1}_{\{e \in C(o)\}}}{\mathbf{P}_{C(o)}(e)} && \text{by (23)} \\
&= \sum_{e \in \Pi} \nu(\varphi : e \in \varphi) \frac{\mathbf{1}_{\{e \in C(o)\}}}{\mathbf{P}_{C(o)}(e)} && \text{by (28) from proposition 89} \\
&= \sum_{e \in \Pi} \int_{\Upsilon(o)} \mathbf{1}_{\{e \in \varphi\}} d\nu(\varphi) \frac{\mathbf{1}_{\{e \in C(o)\}}}{\mathbf{P}_{C(o)}(e)} \\
&= \int_{\Upsilon(o)} \sum_{e \in \Pi} \mathbf{1}_{\{e \in \varphi\}} \frac{\mathbf{1}_{\{e \in C(o)\}}}{\mathbf{P}_{C(o)}(e)} d\nu(\varphi) && \text{by Fubini 52} \\
&= \int_{\Upsilon(o)} \frac{\mathbf{1}_{\{\varphi_\Pi \subseteq C(o)\}}}{\mathbf{P}_{C(o)}(\varphi_\Pi)} d\nu(\varphi) && \text{as } |\Pi \cap \varphi| = 1,
\end{aligned}$$

therefore we have

$$\begin{aligned}
&\mathcal{E}(\nu|_\Pi) \\
&= \mathbf{E}\, \chi(\nu|_\Pi)^2 \\
&= \mathbf{E} \int_{\Upsilon(o)} \frac{\mathbf{1}_{\{\varphi_\Pi \subseteq C(o)\}}}{\mathbf{P}_{C(o)}(\varphi_\Pi)} d\nu(\varphi) \int_{\Upsilon(o)} \frac{\mathbf{1}_{\{\varphi_\Pi \subseteq C(o)\}}}{\mathbf{P}_{C(o)}(\varphi_\Pi)} d\nu(\varphi) \\
&= \mathbf{E} \int_{\Upsilon(o)^2} \frac{\mathbf{1}_{\{\varphi_\Pi \cup \psi_\Pi \subseteq C(o)\}}}{\mathbf{P}_{C(o)}(\varphi_\Pi)\mathbf{P}_{C(o)}(\psi_\Pi)} d\nu \times \nu(\varphi, \psi) && \text{by Fubini 52} \\
&= \int_{\Upsilon(o)^2} \frac{\mathbf{P}_{C(o)}(\varphi_\Pi \cup \psi_\Pi)}{\mathbf{P}_{C(o)}(\varphi_\Pi)\mathbf{P}_{C(o)}(\psi_\Pi)} d\nu \times \nu(\varphi, \psi) && \text{by Fubini 52}
\end{aligned}$$

□

Definition 92 *Let* $\nu \in \mathcal{M}_1(\Upsilon(o))$*, then we say that* ν *has an* **exponential intersection tail** *with parameters* $C \in \mathbb{R}_+, \alpha \in]0,1[$*, or short* $EIT_C(\alpha)$*, if*

$$\forall\, k \in \mathbb{N} : \nu \times \nu((\varphi, \psi) : |\varphi \cap \psi| \geq k) \leq C\alpha^k. \tag{31}$$

Proposition 93 *If $br(T) > 1$ then for every $\lambda \in]1, br(T)[$ and $\alpha \in]\frac{1}{\lambda}, 1[$ there exists $\nu \in \mathcal{M}_1(\Upsilon(o))$ and $C(\lambda, \alpha) \in \mathbb{R}_+$ such that ν has $EIT_{C(\lambda,\alpha)}(\alpha)$.*

PROOF. Let f be a λ-flow on T from o. Regard $g = \frac{f}{St(f)}$ and let ν be its by proposition 88 associated probability measure on $\Upsilon(o)$. Take $\beta \in]1, \lambda[$ and define functions w and c by $\forall\, k \in \mathbb{N} : w(k) = \beta^k \lambda^{-k}$ and $c(k) = \beta^{-k}$. Then we have

$$\forall\, e \in E : f(e) \leq \lambda^{-l_{e(v)}} = w(l_{e(v)})c(l_{e(v)})$$

and

$$W(\lambda, \beta) = \sum_{k\in\mathbb{N}} w(k) = \sum_{k\in\mathbb{N}} \left(\frac{\beta}{\lambda}\right)^k = \frac{\lambda}{\lambda - \beta}.$$

Now we can apply proposition 37 to bound the energy of g

$$St(f)^2 \mathcal{E}(g)_c = \mathcal{E}(f)_c \leq St(f) W(\lambda, \beta)$$

which we use $\forall\, k \in \mathbb{N}$ to show

$$\begin{aligned}
&\nu \times \nu(\{(\varphi, \psi) : |\varphi \cap \psi| \geq k\}) \\
&= \nu \times \nu(\bigcap_{e\in T_k} \{(\varphi, \psi) : |\varphi \cap \psi| \geq k \wedge e \in \varphi \cap \psi\}) \\
&= \sum_{e\in T_k} g^2(e) = \beta^{-k} \sum_{e\in T^k} g^2(e)\beta^k \\
&\leq \beta^{-k}\mathcal{E}(g)_c \leq \frac{W(\lambda, \beta)}{St(f)}\beta^{-k}
\end{aligned}$$

using proposition 37 to conclude with $\alpha = \frac{1}{\beta}$ and $C(\lambda, \alpha) = \frac{W(\lambda,\beta)}{St(f)}$. □

5 Independent percolation on trees

Following $T = (V, E)$ is a leafless, locally finite tree with $br(T) < \infty$.

Definition 94 *Take independent Bernoulli(p) random variables $\{X_e\}_{e \in E}$, then we call this* **independent bond percolation with parameter** *p on T and abbreviate it by $\mathcal{P}_p(E)$. Similarly we denote by $\mathcal{P}_p(V)$ an* **independent site percolation with parameter** *p on T.*

Proposition 95 (Independent percolation bond-site equivalence) *Let $p \in [0, 1]$, then we have*

$$\mathcal{P}_p(E) \textit{ percolates } \Leftrightarrow \mathcal{P}_p(V) \textit{ percolates.}$$

PROOF. Root T at $o \in V$. Now use the canonical vertex-edge bijection 17 to switch between bond and site percolation on the rooted tree and theorem 78 to switch between the rooted and unrooted versions of T. □

So from now on all results will be stated in terms of bond percolation, although they can be easily transferred to site percolation via proposition 95. Following Kolmogorov's $0 - 1$ law for 0-independent percolation 70 we know that $\mathbf{P}(\mathcal{P}_p(E) \text{ percolates}) \in \{0, 1\}$ and it's also evident that this probability is rising in p as independent percolation is a monotone property of graphs. To show this in detail, create for $p < p'$ two independent bond percolations $\mathcal{P}_p(E)$ $\{X_e\}_{e \in E}$ and $\mathcal{P}_{p'}(E)$ $\{X'_e\}_{e \in E}$, where for each edge the Bernoulli random variables have been coupled as detailled in section 3.2. Thus we have $\{\mathcal{P}_{p'}(E) \text{ percolates}\} \subseteq \{\mathcal{P}_p(E) \text{ percolates}\}$. As $\mathbf{P}(\mathcal{P}_0(E) \text{ percolates}) = 0$ and $\mathbf{P}(\mathcal{P}_1(E) \text{ percolates}) = 1$ we are interested in the following:

Definition 96 *Let T be an infinite, locally finite, leafless tree. Take independent bond percolation. Define the* **critical probability** *as*

$$p_c(T) = sup\{p \in [0, 1] : \mathcal{P}_p(E) \textit{ doesn't percolate}\}. \tag{32}$$

Notice that we have $p^0_{\min}(T) = p_c(T) = p^0_{\max}(T)$ by the above coupling argument. This sudden shift in behaviour at $p_c(T)$ is also called a **phase transition**. Similar behaviour on finite random graphs can be observed for example on the $G(n, p)$ model by Erdös and Rènyi introduced in 1960 in [4], for a summary of some results see [13].

Corollary 97 *In the case of independent percolation statement* (19a) *from theorem 78 can be changed to*

$$(\forall\, v \in V : \xi_v > 0) \Leftrightarrow \mathbf{P}(\mathcal{P} \textit{ percolates on } T) = 1 \tag{33}$$

Proof. Remember that we have rooted T at o fixed, but arbitrarily chosen beforehand. We now know that $\exists\, v \in V : \xi_v > 0$ and we show that this induces $\xi_o > 0$. As we can write for all $v \in V$

$$\xi_o \geq \mathbf{P}\,(o \leftrightarrow v \leftrightarrow \infty) = \mathbf{P}\,(o \leftrightarrow v | v \leftrightarrow \infty)\xi_v = p^{l_v}\xi_v > 0,$$

we have proven (33). □

5.1 Determining $p_c(T)$

We now proceed to determine the above quantity. The proof follows the structure of the proof given in [9], chapter 4, by Lyons and Peres, prunning everything related to random walks and network theory, as it is not a goal to detail the connections to these fields (see [9], chapter 2 or [10] for an introduction to those fields), but to present a self-contained proof within the scope of this work. First we use the 1^{st} moment method found in proposition 81 to minorize $p_c(T)$:

Proposition 98 (1^{st} moment method) *We have*

$$\mathcal{P}_p(E) \text{ percolates} \Rightarrow \exists\, \frac{1}{p}\text{-branching flow on } T, \tag{34a}$$

$$\nexists\, \frac{1}{p}\text{-branching flow on } T \Rightarrow \mathcal{P}_p(E) \text{ doesn't percolate.} \tag{34b}$$

Proof. Root T at o. As we have a $\mathcal{P}_p(E)$ we have $\mathbf{P}\,(e \in C(o)) = p^{l_{v(e)}}$ and by the 1^{st} moment method 81 thus

$$\xi_o = \mathbf{P}\,(o \leftrightarrow \infty) \leq \inf_{\Pi \in \Pi(o)} \sum_{e \in \Pi} \left(\frac{1}{p}\right)^{-l_{v(e)}}. \tag{35}$$

Observe that the right-hand side of (35) is positive iff there is a $\frac{1}{p}$-branching flow on T (by the secondary definition of $br(T)$ in (6b)). □

Lemma 99 (p_c-Minoration)

$$p_c(T) \geq \frac{1}{br(T)}.$$

Proof. Now take $p < \frac{1}{br(T)} \Leftrightarrow \frac{1}{p} > br(T)$, therefore no $\frac{1}{p}$-branching flow exists. Proposition 98 now asserts that for all choices of a root o we have $\xi_o = 0$ and therefore $\mathcal{P}_p(E)$ doesn't percolate by (19b). □

The majoration is effected by means of the 2^{nd} moment method and exponential intersection tails, deviating from the proof found in [9] and using the methods in [11], section 10.

Proposition 100 *In the case of* $\mathcal{P}_p(E)$ *and* $\Pi \in \Pi(o)$ *the path percolation kernel from 90 has the form:* $K_\Pi(\varphi, \psi) = p^{-|\varphi_\Pi \cap \psi_\Pi|}$.

PROOF.

$$K_\Pi(\varphi, \psi) = \frac{\mathbf{P}_{C(o)}(\varphi_\Pi \cup \psi_\Pi)}{\mathbf{P}_{C(o)}(\varphi_\Pi)\mathbf{P}_{C(o)}(\psi_\Pi)} = \frac{p^{|\varphi_\Pi \cup \psi_\Pi|}}{p^{|\varphi_\Pi|}p^{|\psi_\Pi|}}$$
$$= \frac{p^{|\varphi_\Pi \cap \psi_\Pi| + |\varphi_\Pi \setminus \psi_\Pi| + |\psi_\Pi \setminus \varphi_\Pi|}}{p^{|\varphi_\Pi \cap \psi_\Pi| + |\varphi_\Pi \setminus \psi_\Pi| + |\varphi_\Pi \cap \psi_\Pi| + |\psi_\Pi \setminus \varphi_\Pi|}} = p^{-|\varphi_\Pi \cap \psi_\Pi|}$$

□

Lemma 101

$$p_c(T) \leq \frac{1}{br(T)}.$$

PROOF. For $p \in]\frac{1}{br(T)}, 1]$ take $\lambda \in]\frac{1}{p}, br(T)[$. Call $\nu \in \mathcal{M}_1(\Upsilon(o))$ the probability measure guaranteed by proposition 93 to have $EIT_{C(\lambda,\alpha)}(\alpha)$ where $\alpha \in]\frac{1}{\lambda}, p[$. Now for $\Pi \in \Pi(o)$ let $\nu|_\Pi \in \mathcal{M}_1(\Pi)$ be the projection of ν on Π by proposition 89, hence we have

$$\begin{aligned}
\mathcal{E}(\nu|_\Pi) &= \int_{\Upsilon(o)^2} K_\Pi(\varphi, \psi) d\nu \times \nu(\varphi, \psi) && \text{by lemma 91} \\
&= \int_{\Upsilon(o)^2} p^{-|\varphi_\Pi \cap \psi_\Pi|} d\nu \times \nu(\varphi, \psi) && \text{by proposition 100} \\
&\leq \int_{\Upsilon(o)^2} p^{-|\varphi \cap \psi|} d\nu \times \nu(\varphi, \psi) \\
&= \sum_{k=0}^{\infty} \int_{\Upsilon(o)^2} p^{-k} \mathbf{1}_{\{|\varphi \cap \psi| = k\}} d\nu \times \nu(\varphi, \psi) && \text{by Fubini 52} \\
&= \sum_{k=0}^{\infty} p^{-k} \nu \times \nu(\{(\varphi, \psi) : |\varphi \cap \psi| = k\}) \\
&\leq \sum_{k=0}^{\infty} p^{-k} C(\lambda, \alpha) \alpha^k && \text{by } EIT_{C(\lambda,\alpha)}(\alpha) \\
&= C(\lambda, \alpha) \sum_{k=0}^{\infty} \left(\frac{\alpha}{p}\right) = C(\lambda, \alpha) \frac{\alpha}{p - \alpha},
\end{aligned}$$

thus by the second moment method from lemma 86:

$$\xi_o \geq \inf_{\Pi \in \Pi(o)} \sup_{\mu \in \mathcal{M}_1(\Pi)} \frac{1}{\mathcal{E}(\mu)} \geq \inf_{\Pi \in \Pi(o)} \frac{1}{\mathcal{E}(\nu|_\Pi)} \geq \frac{p - \alpha}{\alpha C(\lambda, \alpha)} > 0.$$

Apply theorem 78 to conclude that $\mathcal{P}_p(E)$ percolates. $\square$

And finally we put it all together:

Theorem 102 (Critical value of p**)** *Let T be an infinite, locally finite, leafless tree with $br(T) < \infty$. Then*

$$p_c(T) = \frac{1}{br(T)}. \tag{36}$$

PROOF. Combine lemmata 101 and 99. $\square$

5.2 Behaviour at $p_c(T)$

We call independent percolation on T at the critical value $\frac{1}{br(T)}$ **critical percolation**. It is not possible to determine its behaviour just in terms of the existence of an $br(T)$-branching flow, but nevertheless some statements are possible. For exact determination by means of other methods see the comments before lemma 104.

Corollary 103 *We have*

$$\mathcal{P}_{\frac{1}{br(T)}}(E) \text{ percolates} \Rightarrow \exists\, br(T)\text{-branching flow on } T, \tag{37a}$$

$$\nexists\, br(T)\text{-branching flow on } T \Rightarrow \mathcal{P}_{\frac{1}{br(T)}}(E) \text{ doesn't percolate.} \tag{37b}$$

PROOF. This is a simple corollary of the 1^{st} moment method from 98. $\square$

This leaves open the question under which conditions a $\frac{1}{br(T)}$-flow results in $\mathcal{P}_{\frac{1}{br(T)}}(E)$ percolating. The following lemma, taken from [9], chapter 4.4, pp 111–112, shows how in the case of independent percolation one can reverse the 2^{nd} moment inequality by a constant factor, which will allow further analysis of the problem. Together with network theory (see [9], chapter 2 for a treatment of this topic) this would allow the complete specification. As this would go beyond the scope of this work we have to fall back on additional properties of T.

Lemma 104 (2^{nd} moment method reversed)

$$\xi_o = \mathbf{P}\left(o \leftrightarrow \infty\right) \leq 2 \inf_{\Pi \in \Pi(o)} \sup_{\mu \in \mathcal{M}_1(\Pi)} \frac{1}{\mathcal{E}(\mu)}. \tag{38}$$

PROOF. Take any cutset $\Pi \in \Pi(o)$ and impose a linear ordering $\prec$ on Π. It always exists, as Π is finite: take for example the ordering from left to right (remember that T is oriented downwards so Π is cutting horizontally). Define the random variable

$$e^\star : \quad \Omega \to \Pi \cup \{\Delta\} \quad \omega \mapsto \begin{cases} e & \min\{e \in \Pi : o \leftrightarrow e\} \text{ if } \{o \leftrightarrow \Pi\} \\ \Delta & \{o \not\leftrightarrow \Pi\}. \end{cases}$$

We now define the (possible defective when $e^\star = \Delta$) hitting measure

$$\sigma(e) = \mathbf{P}(e^\star = e)$$

and as $\mathbf{P}(o \leftrightarrow \Pi) \geq \mathbf{P}(o \leftrightarrow e) = p^{l_{v(e)}} > 0$ we can define

$$\mu = \frac{\sigma}{\mathbf{P}(o \leftrightarrow \Pi)} \in \mathcal{M}_1(\Pi).$$

Now we have for all $e \in E$

$$\begin{aligned} &\sum_{f \preceq e} \sigma(f) \frac{\mathbf{P}(o \leftrightarrow f, o \leftrightarrow e)}{\mathbf{P}(o \leftrightarrow f)} \\ &= \sum_{f \preceq e} \mathbf{P}(e^\star = f) \mathbf{P}(o \leftrightarrow e | o \leftrightarrow f) \\ &= \sum_{f \preceq e} \mathbf{P}(e^\star = f) \mathbf{P}(o \leftrightarrow e | e^\star = f) \\ &= \sum_{f \in \Pi} \mathbf{P}(e^\star = f) \mathbf{P}(o \leftrightarrow e | e^\star = f) \qquad e \prec f \Rightarrow \mathbf{P}(o \leftrightarrow e | e^\star = f) = 0 \\ &= \mathbf{P}(o \leftrightarrow e) \end{aligned}$$

thus we have

$$\begin{aligned} &\mathcal{E}(\mu) \\ &= \sum_{e,f \in E} \mu(e)\mu(f)\kappa_\Pi(e,f) \\ &\leq \sum_{e \in E} 2\mu(e) \sum_{f \preceq e} \mu(f)\kappa_\Pi(e,f) && \text{by symmetry} \\ &= \sum_{e \in E} \frac{2\mu(e)}{\mathbf{P}(o \leftrightarrow \Pi)\mathbf{P}(o \leftrightarrow e)} \sum_{f \preceq e} \sigma(f) \frac{\mathbf{P}(o \leftrightarrow f, o \leftrightarrow e)}{\mathbf{P}(o \leftrightarrow f)} \\ &= \sum_{e \in E} \frac{2\mu(e)\mathbf{P}(o \leftrightarrow e)}{\mathbf{P}(o \leftrightarrow \Pi)\mathbf{P}(o \leftrightarrow e)} && \text{previous calculation} \\ &= \frac{2}{\mathbf{P}(o \leftrightarrow \Pi)} \sum_{e \in E} \mu(e) = \frac{2}{\mathbf{P}(o \leftrightarrow \Pi)}. \end{aligned}$$

Reversing this yields

$$\mathbf{P}(o \leftrightarrow \Pi) \leq \frac{2}{\mathcal{E}(\mu)} \leq 2 \sup_{\mu \in \mathcal{M}_1(\Pi)} \frac{1}{\mathcal{E}(\mu)}$$

and finally, using (18b), we arrive at

$$\mathbf{P}(o \leftrightarrow \infty) = \inf_{\Pi \in \Pi(o)} \xi_o^{\Pi} \leq 2 \inf_{\Pi \in \Pi(o)} \sup_{\mu \in \mathcal{M}_1(\Pi)} \frac{1}{\mathcal{E}(\mu)}.$$

□

Proposition 105 (Measure-Flow Bijection) *We have* $\forall\, o \in V : \forall\, \Pi \in \Pi(o) : \forall\, \mu \in \mathcal{M}_1(\Pi)$*: There exists a unique nonnegative unit flow* f *with strength* 1 *on* $T|_{\Pi}$ *for which the following equation holds:*

$$\mathcal{E}(\mu) = 1 + \mathcal{E}(f)_c, \tag{39}$$

where $\mathcal{E}(f)_c$ *is calculated with respect to the function* $c(e) = \frac{p^{l_{v(e)}}}{1-p}$.

Proof. It follows from proposition 33 that every flow on $T|_{\Pi}$ is determined by its values on $\partial T|_{\Pi} = v(\Pi)$ which is isomorph to Π, thus the bijection between measures and nonnegative flows of strength 1. Finally the relation

(39) between the energies is shown by

$$
\begin{aligned}
&\mathcal{E}(\mu)\\
&= \sum_{v,w \in v(\Pi)} \frac{\mu(e(v))\mu(e(w))}{\mathbf{P}\left(o \leftrightarrow v \wedge w\right)}\\
&= \sum_{v,w \in v(\Pi)} \frac{\mu(e(v))\mu(e(w))}{p^{l_{v\wedge w}}} \qquad \text{by independence of the percolation}\\
&= \sum_{v,w \in v(\Pi)} \mu(e(v))\mu(e(w))(1 + p^{-l_{v\wedge w}} - 1)\\
&= \sum_{e,f \in \Pi} \mu(e)\mu(f) + \sum_{v,w \in v(\Pi)} \mu(e(v))\mu(e(w))(p^{-l_{v\wedge w}} - 1)\\
&= 1 + \sum_{v,w \in v(\Pi)} f(v)f(w)(p^{-l_{v\wedge w}} - 1)\\
&= 1 + \sum_{v,w \in v(\Pi)} f(v)f(w) \sum_{e \in E|_\Pi :\, v(e) \leq v \wedge w} (1-p)p^{-l_{v(e)}}\\
&= 1 + \sum_{e \in E|_\Pi} (1-p)p^{-l_{v(e)}} \sum_{v,w \in v(\Pi) :\, v(e) \leq v\wedge w} f(v)f(w)\\
&= 1 + \sum_{e \in E|_\Pi} (1-p)p^{-l_{v(e)}} \sum_{v \in v(\Pi) :\, v(e) \leq v} f(v) \sum_{w \in v(\Pi) :\, v(e) \leq w} f(w)\\
&= 1 + \sum_{e \in E|_\Pi} (1-p)p^{-l_{v(e)}} f(v(e))^2 \qquad \text{by proposition 33}\\
&= 1 + \mathcal{E}(f)_c
\end{aligned}
$$

□

Lemma 106 *If $gr(T) = br(T)$ and $br(T) > 1$ then $\mathcal{P}_{\frac{1}{br(T)}}(E)$ doesn't percolate.*

Proof. Root T at o. First we see that, as $gr(T) = br(T)$, we can apply corollary 50 to get $C \in \mathbb{R}_+ : \forall n \in \mathbb{N} : |T^n| \leq C(br(T))^n$. Next we minorate $\mathcal{E}(f)_c$ for any nonnegative unit flow f on $T|_n$ and $c(e) = \frac{p^{l_{v(e)}}}{1-p}$ from proposition 105:

$$\begin{aligned}
&\mathcal{E}(f)_c \\
&= \sum_{e\in E|_n} (1-p)p^{-l_{v(e)}} f(e)^2 \\
&\geq \sum_{k=1}^{n} \left(\sum_{e\in T^k} \frac{p^k}{1-p} \right)^{-1} && \text{by Nash-Williams 38} \\
&\geq (1-p) \sum_{k=1}^{n} \frac{1}{C(br(T))^k p^k} \\
&= (1 - \frac{1}{br(T)}) \sum_{k=1}^{n} \frac{1}{C} && \text{as } p = \frac{1}{br(T)} < 1 \\
&= \frac{n(1-\frac{1}{br(T)})}{C}.
\end{aligned}$$

Next we majorate

$$\begin{aligned}
&\xi_o \\
&\leq 2 \inf_{\Pi\in\Pi(o)} \sup_{\mu\in\mathcal{M}_1(\Pi)} \frac{1}{\mathcal{E}(\mu)} && \text{by reversal } 2^{nd} \text{ moment method 104} \\
&= 2 \inf_{\Pi\in\Pi(o)} \sup_{f \text{ on } T|_\Pi} \frac{1}{1+\mathcal{E}(f)_c} && \text{by proposition 105} \\
&\leq 2 \inf_{n\in\mathbb{N}} \sup_{f \text{ on } T|_n} \frac{1}{1+\mathcal{E}(f)_c} && \text{taking only certain cutsets} \\
&\leq 2 \inf_{n\in\mathbb{N}} \sup_{f \text{ on } T|_n} \frac{C}{1+n(1-\frac{1}{br(T)})} && \text{by the previous calculation} \\
&\leq 2 \inf_{n\in\mathbb{N}} \frac{C}{1+n(1-\frac{1}{br(T)})} && \text{independent of } f \\
&= 0
\end{aligned}$$

As the choice of o was independent of $\mathcal{P}_{\frac{1}{br(T)}}(E)$ we see by (19b) that $\mathcal{P}_{\frac{1}{br(T)}}(E)$ doesn't percolate. □

Corollary 107 *If T is (sup-/super-) periodic and $br(T) > 1$ then $\mathcal{P}_{\frac{1}{br(T)}}(E)$ doesn't percolate.*

PROOF. By theorems 48 and 49 any (sup-/super-) peridioc tree T has $gr(T) = br(T)$, hence we can apply lemma 106. □

5.3 Branching number of the giant component

Independent percolation $\mathcal{P}_p(E)$ on T has the effect of splitting up T into a forest $F_p(\omega) = (V, E_p(\omega))$ of smaller trees. We are now interested in the structure and especially the branching number of the infinite trees in $F_p(\omega)$ in the case of $\mathcal{P}_{\frac{1}{br(T)}}(E)$ percolating.

Lemma 108 *Take $p, p' \in]0,1[$. Regard the percolations $\{Y_e^p\}_{e\in E}$, which is a $\mathcal{P}_p(E)$, and $\{Y_e^{pp'}\}_{e\in E}$, which is a $\mathcal{P}_{pp'}(E)$. Furthermore have a percolation $\mathcal{P}_{p'}(E_p(\omega))$ $\{Y_e^{p'}(\omega)\}_{e\in E_p(\omega)}$ conditionally defined on $F_p(\omega)$. Then the laws of $\{Y_e^{p'}(\omega)\}_{e\in E_p(\omega)}$ and $\{Y_e^{pp'}\}_{e\in E}$ conditioned on $F_p(\omega)$ are almost surely the same.*

Proof. We can assume that $\{Y_e^p\}_{e\in E}$ and $\{Y_e^{pp'}\}_{e\in E}$ are coupled as described in proposition 56. For $e \in E$ and $\omega : e \in F_p(\omega)$ we have

$$\begin{aligned}\mathbf{P}\left(Y_e^{p'}(\omega) = 1 | e \in F_p(\omega)\right) = p' = \frac{pp'}{p} = \frac{\mathbf{P}\left(Y_e^{pp'} = 1\right)}{\mathbf{P}\left(Y_e^p = 1\right)} \\ = \frac{\mathbf{P}\left(Y_e^{pp'} = 1, Y_e^p = 1\right)}{\mathbf{P}\left(Y_e^p = 1\right)} = \mathbf{P}\left(Y_e^{pp'} = 1 | Y_e^p = 1\right). \quad (40)\end{aligned}$$

Note that $\{\omega : e \in F_p(\omega)\} = \{\omega : Y_e^p = 1\}$. The calculation in (40) can easily be adapted to the case $Y_e^{p'} = 0$ and extended to a finite number of edges. Therefore the conditional laws coincide on the product σ-algebra over E almost surely and also over the tail σ-algebra on E almost surely (see also figure 3 on 39). □

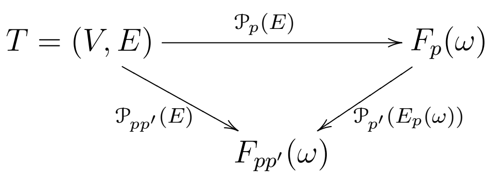

Figure 3: Communativity of iterated independent percolation with parameters p and p' with independent percolation with parameter pp' on T.

Theorem 109 *Take p such that $\mathcal{P}_p(E)$ percolates. Then there exists an infinite tree $T'(\omega)$ in $F_p(\omega)$ such that*

$$p_c(T'(\omega)) = \frac{p_c(T)}{p} \text{ almost surely.} \quad (41)$$

Proof. Take the biggest (with respect to its branching number) infinite tree $T'(\omega)$ in $F_p(\omega)$, which exists almost surely as $\mathcal{P}_p(E)$ percolates. Define a percolation $\mathcal{P}_{p'}(E_p(\omega))$ $\{Y_e^{p'}(\omega)\}_{e\in E_p(\omega)}$ conditionally on $F_p(\omega)$. As we investigate the percolation behaviour of this second percolation, we see that, conditional on $T'(\omega)$, by lemma 108, this has the same law as a $\mathcal{P}_{pp'}(E)$, which will percolate if $pp' > \frac{1}{br(T)}$ and not if $pp' < \frac{1}{br(T)}$, hence $p_c(T'(\omega)) = \frac{p_c(T)}{p}$ almost surely. □

Corollary 110 *Take p such that $\mathcal{P}_p(E)$ percolates. Then there exists an infinite tree $T'(\omega)$ in $F_p(\omega)$ such that*

$$br(T'(\omega)) = p\, br(T) \text{ almost surely.} \tag{42}$$

Proof. Use theorem 109 in conjunction with theorem 102. □

6 1-independent percolation on trees

This section has been worked out after the notes [2] taken by Pierre Mathieu of a presentation given by Bollobas on the topic. Another use of 1-independent percolation, this time on $\mathbb{Z}^2$, has taken place by Bollobas and Riordan in [3] to prove results about independent percolation on $\mathbb{Z}^2$.
In the following $p \in]0,1[$ and $q = 1-p$. $T = (V, E)$ is as usually an infinite, locally finite, leafless tree with $br(T) < \infty$. If necessary let T be rooted at o. We will use the notation established in definitions 73 and 74. For any $w \in V$ we will denote the ranks of the respective nodes by

$$
\begin{aligned}
&r = r_w && \\
&r_i = r_{w_i} && \text{for the children } \{w_i\}_{i=1}^{r} \text{ of } w \\
&r_{i,j} = r_{w_{i,j}} && \forall\, i \in \{1,\dots,r\}\text{: for the children } \{w_{i,j}\}_{j=1}^{r_i} \text{ of } w_i.
\end{aligned}
$$

6.1 Analytical foreplay

We start by looking at two functions and a sequence they generate.

Proposition 111 *Regard*

$$f_p : \quad]0,1] \to \mathbb{R} \qquad x \mapsto 1 - \frac{q}{x} \tag{43}$$

then for $p \in [\frac{3}{4}, 1]$ the function f_p has an unique positive fixed point

$$c(p) = \frac{1+\sqrt{1-4q}}{2} \tag{44}$$

lying in $[\frac{1}{2}, 1]$. This can be reversed bijectively to yield (see (48))

$$p(c) = g\left(\frac{1}{c}\right) = c^2 - c + 1. \tag{45}$$

Proof. First we start with some properties of f_p

$$
\begin{aligned}
{f_p}'(x) = \frac{q}{x^2} > 0 \qquad {f_p}''(x) = -\frac{2q}{x^3} < 0 \\
\lim_{x\to 0} f_p(x) = -\infty \qquad \lim_{x\to 1} f_p(x) = p
\end{aligned} \tag{46}
$$

Next we resolve the fixed point equation

$$c = f_p(c) = 1 - \frac{q}{c} \Leftrightarrow c^2 - c + q = 0 \tag{47}$$

to arrive at (44) and (45). We deduce the bijection from the positive first derivatives of the following two expressions

$$c(p)' = \frac{2}{\sqrt{4p-3}} > 0 \text{ iff } p > \frac{3}{4} \quad \text{and} \quad p(c)' = 2c - 1 > 0 \text{ iff } c > \frac{1}{2}.$$

□

Proposition 112 *The function*

$$g : \quad [1, \infty[\to]0, 1[\qquad y \mapsto 1 - \frac{y-1}{y^2} \tag{48}$$

has its global minimum at 2 with value $\frac{3}{4}$ *and there is a bijection between* $g(]1,2[)$ *and* $g(]2,\infty[)$*. Furthermore* g *is strictly decreasing in* $[1,2]$ *and strictly increasing in* $[2,\infty]$*.*

PROOF. We take note of the following facts concerning g

$$\begin{aligned} g'(y) &= \frac{y-2}{y^3} & g''(y) &= \frac{6-2y}{y^4} \\ \lim_{y\to 1} g(y) &= 1 & \lim_{y\to\infty} g(y) &= 1 \end{aligned} \tag{49}$$

from which we can easily deduce the statements about g. □

Proposition 113 *For any* $x \in [1,2[$ *and* $p, p' \in [\frac{3}{4}, 1]$ *we have*

$$p' < g(y) < p \Leftrightarrow c(p') < \frac{1}{y} < c(p) \tag{50}$$

PROOF.

$$\begin{aligned} & p > g(y) > p' \geq \frac{3}{4} && \text{by proposition 112} \\ \Leftrightarrow\ & g\left(\frac{1}{c(p)}\right) > g(y) > g\left(\frac{1}{c(p')}\right) \geq \frac{3}{4} && \text{by (47)} \\ \Leftrightarrow\ & \frac{1}{c(p)} < y < \frac{1}{c(p')} && \text{by proposition 112} \\ \Leftrightarrow\ & c(p) > \frac{1}{y} > c(p') \end{aligned}$$

□

Proposition 114 *For $p \in [0,1]$ define the sequence $\{x_k\}_{k\in\mathbb{N}}$ by*

$$x_k = \begin{cases} 1 & k = 0 \\ f_p(x_{k-1}) & k \geq 1 \wedge x_{k-1} > q \\ 0 & \text{otherwise}, \end{cases} \tag{51}$$

where f_p defined as in (43). It is monotonically decreasing and for $p \geq \frac{3}{4}$ strictly decreasing towards $c(p)$.

Proof. If $p \geq \frac{3}{4}$ the monotonicity of f_p implies the monotonicity of x_k:

$$\begin{aligned} & x > c(p) \\ \Leftrightarrow\ & x[c(p) - 1] > c(p)[1 - c(p)] = q && \text{by (45)} \\ \Leftrightarrow\ & f(x) = 1 - \frac{q}{x} > c(p) && \text{by (46).} \end{aligned}$$

□

6.2 The canonical model

In this section we present the canonical 1-independent bond percolation with parameter p. The importance it plays will become appearent in section 6.3, where it is shown that it is in a certain sense the best 1-independent bond percolation with parameter p existing on a given rooted tree T.

Model 115 **Finite case***: Let $p \in [0,1]$. Take a finite tree T rooted at o with N full levels. Define a sequence $\{c_k\}_{k=1}^{N}$ by $c_k = x_{N-k}$, where $\{x_k\}_{k\in\mathbb{N}}$ as in 114. Take independent Bernoulli site percolation $\{X_v\}_{v\in V}$ on T with parameter c_k on T_k. Define the* **canonical 1-independent bond percolation** *$\{Y_e\}_{e\in E}$ as follows:*

$$\{Y_e = Y_{e(v)} = 0\} = \{X_v = 1 \wedge X_{p(v)} = 0\} \tag{52}$$

Infinite case*: Let $p \in [\frac{3}{4}, 1]$ and root T at o. Take a $\mathcal{P}_{c(p)}(V)$ percolation $\{X_v\}_{v\in V}$. Define the canonical 1-independent bond percolation with parameter p on T, abbreviated as $\mathcal{P}_p^{1,can}(E)$, $\{Y_e\}_{e\in E}$ as in (52).*

Definition 116 *In the case of canonical 1-independent bond percolation we call the* **underlying path** *of a downpath from a node w to a node v the set of nodes which are endpoints of edges on the downpath. Likewise the definition is extended to* **underlying ray** *in the obvious sense. This allows to play the $\mathcal{P}_p^{1,can}(E)$ back onto its underlying $\mathcal{P}_{c(p)}(V)$ and analogous in the finite case.*

Proposition 117 *We want to classify the* **structure of the underlying paths/rays** *in the canonical* 1*-independent bond percolation. In the finite case the underlying path consists of only* 1*'s. In the infinite case there are three possibilites for the underlying ray:*

$$0-0-0-0-0-0-0-0-\dots \qquad \Rightarrow \textit{infinite } 0\textit{-ray} \tag{53a}$$
$$\underbrace{1-1-\dots\dots-1-1}_{\textit{finitely many}}-0-0-\dots \qquad \Rightarrow \textit{infinite } 0\textit{-ray} \tag{53b}$$
$$1-1-1-1-1-1-1-1-\dots \qquad \Rightarrow \textit{infinite } 1\textit{-ray.} \tag{53c}$$

Proof. By the definition of the canonical 1-independent bond percolation in 52 an open edge implies that in its underlying 2 endnodes the case $0-1$ can't appear. In the case of finite N the path could contain only 1's, because if it would contain a 0, it could only be followed by other 0's and would encounter latest at level N a 1, thereby closing at least one edge along the path. In the infinite case it is possible, that once switched from 1's to 0's to stay with them, hence the three above possibilities. □

Notation 118 *Following the notation established in definition 73 we write for the* $\mathcal{P}_p^{1,can}(E)$

$O_w^{m,can(N)}$	*downpath to level* m *from* w *in the* N*-finite case*
$O_w^{m,can(\infty)}$	*downpath to level* m *from* w *in the infinite case*
$O_w^{can(\infty)}$	*downray from* w *for* $\mathcal{P}_p^{1,can}(E)$
$O_w^{can(\infty,1)}$	*downray from* w *based on a* 1*-ray in the underlying* $\mathcal{P}_{c(p)}(V)$

This adaption of notation is also carried over to the events and probabilites defined in 74.

Lemma 119 (Canonical convergence) *Let* $p \in [\frac{3}{4}, 1]$ *and* T *be rooted at* o*. For* $w \in V$ *we have*

$$O_w^{N,can(N)} \xrightarrow[N\to\infty]{\mathbf{P}} O_w^{can(\infty,1)} \tag{54}$$

Proof. We start by creating a sequence of percolations which are coupled to each other. First assign an independent family of Unif([0, 1])-distributed random variables to the vertices of T: $\{U_v\}_{v\in V}$ i.i.d. Unif([0, 1]). For $N \in \mathbb{N}$ we create now a finite canonical 1-independent bond percolation $\mathcal{P}_p^{1,can(N)}(E(T|_N))$ by coupling the underlying independent site percolation on $V(T|_N)$ on $\{U_v\}_{v\in V}$ and setting all other vertices to 1. Finally we create an infinite canonical 1-independent bond percolation $\mathcal{P}_p^{1,can}(E)$ where the underlying $\mathcal{P}_{c(p)}(V)$ is again coupled to the $\{U_v\}_{v\in V}$.

We simplify our reasoning considerably by extending each of the finite percolations $\mathcal{P}_p^{1,can(N)}(E(T|_N))$ by 1's to the rest of T, thereby assuring that all percolations live on E. Next we note the simple fact that

$$\forall\, N \in \mathbb{N}, \forall\, m \geq N : O_w^{m,can(N)} = O_w^{N,can(N)} \tag{55}$$

as a path down to level N is going to ∞ in the extension by 1's to T (see proposition 117). Using this fact and proposition 75 we arrive at

$$\forall\, N \in \mathbb{N} : O_w^{N,can(N)} = O_w^{can(N)}. \tag{56}$$

We also see that for finite N, the paths going through the percolation above level N can only have underlying paths of 1's, as any path containing a 0 could only go on with further 0's and would latest terminate at level N, where we have deterministically only 1's (see (52)). Next we remark that the sequence of events $\{O_w^{can(N)}\}_{N\in\mathbb{N}}$ is monotone decreasing, as

$$O_w^{can(N+1)} = O_w^{N+1,can(N+1)} \subset O_w^{N,can(N+1)} \subset O_w^{N,can(N)} = O_w^{can(N)}. \tag{57}$$

The parameters of the underlying Bernoulli variables on $V(T|_m)$ are coupled in a strictly monotone order by the definition of the sequence c_k (see proposition 114), which implies a coupling of the underlying paths of 1's and leads leads to the following chain:

$$O_w^{m,can(\infty,1)} \subset \ldots \subset O_w^{m,can(N+1)} \subset O_w^{m,can(N)} \subset \ldots.$$

Now for finite m we have only finitely many coupled variables on $V(T|_m)$ and convergence of the sequence c_k (see proposition 114) gives us convergence in probability and we arrive at the core argument of this proof:

$$\forall\, m \geq l_w : O_w^{m,can(N)} \xrightarrow[N\to\infty]{\mathbf{P}} O_w^{m,can(\infty,1)}. \tag{58}$$

Now putting it all together we have:

$$\bigcap_{N=1}^{\infty} O_w^{m,can(N)} = O_w^{m,can(\infty,1)} \qquad \text{using (58)}$$

$$\bigcap_{m=1}^{\infty} \bigcap_{N\geq m}^{\infty} O_w^{m,can(N)} = \bigcap_{k=m}^{\infty} O_w^{m,can(\infty,1)}$$

$$\bigcap_{N=1}^{\infty} \bigcap_{m=1}^{N} O_w^{m,can(N)} = O_w^{can(\infty,1)} \qquad \text{using proposition 75}$$

$$\bigcap_{N=1}^{\infty} \bigcap_{m=1}^{\infty} O_w^{m,can(N)} = O_w^{can(\infty,1)} \qquad \text{using (56)}$$

$$\bigcap_{N=1}^{\infty} O_w^{can(N)} = O_w^{can(\infty,1)} \qquad \text{again using proposition 75}$$

and using the fact that $\{O_w^{can(N)}\}_{N\in\mathbb{N}}$ is a decreasing sequence (see (57)) we arrive at (54). □

Proposition 120 *Let $p \in]\frac{3}{4}, 1]$ and T be rooted at o. If $br(T) < 2$ then*

$$\forall\, w \in V : O_w^{can(\infty,1)} = O_w^{can(\infty)} \textit{almost surely.} \tag{59}$$

Proof. Take $\mathcal{P}_p^{1,can}(E)$ and suppose that it percolates, hence there is a node w with finite level such that there is a downpath from w to ∞. Using the structure of the underlying ray as established in proposition 117 we get for the underlying $\mathcal{P}_{c(p)}(V)$

$$\text{(53a) or (53b)} \Rightarrow \text{infinite 0-cluster} \tag{60a}$$

$$\text{(53c)} \Rightarrow \text{infinite 1-cluster.} \tag{60b}$$

But an infinite 0-cluster as in (60a) would imply a cluster of 1's in $\mathcal{P}_{1-c(p)}(V)$, which by theorem 102 only percolates iff $\frac{1}{br(T)} < 1-c(p)$. But by proposition 113 we get $\frac{1}{2} < c(p) < \frac{1}{br(T)}$, hence we have $\frac{1}{br(T)} < \frac{1}{2} < 1 - c(p)$ and don't percolate. Thus the only chance is to have an underlying 1-ray for a downray from w for $\mathcal{P}_p^{1,can}(E)$. □

6.3 Lower bounds by the canonical model

Proposition 121 *Let T be rooted at o and let $w \in V$, with neither w nor one of its children being leaves. Then the following* **partition** *holds*

$$F_w = H \uplus \biguplus_{i=1}^{r} \left(F_w \cap H^c \cap \bigcap_{k=i+1}^{r} G_k^c \cap G_i \right) \tag{61}$$

where $G_i = \bigcap_{k=1}^{i-1} F_{w_k} \cap O_{w_i}$ *and* $H = \bigcap_{i=1}^{r} F_{w_i}$.

Proof. The key point is the fact that we divide the probability space by the lowest index of a child of w having an open downray.

$$
\begin{aligned}
& F_w \\
&= (F_w \cap H) \uplus (F_w \cap H^c) \\
&= H \uplus (F_w \cap H^c \cap G_r) \uplus (F_w \cap H^c \cap G_r^c) && \text{as } H \subset F_w \\
&= H \uplus \biguplus_{i=1}^{r} \left(F_w \cap H^c \cap \bigcap_{j=i+1}^{r} G_j^c \cap G_i \right) && \text{by induction over } i \\
&= H \uplus \biguplus_{i=1}^{r} \left(F_w \cap H^c \cap \bigcap_{k=i+1}^{r} G_k^c \cap G_i \right)
\end{aligned}
$$

□

Lemma 122 (η-majoration) *Let T be rooted at o and let $w \in V$, with neither w nor one of its children being leaves. For any $\mathcal{P}_p^1(E)$ the following inequality holds*

$$\eta_w \leq q \sum_{i=1}^{r} \left[\left(1 - \prod_{j=1}^{r_i} \eta_{w_{i,j}} \right) \prod_{k=1}^{i-1} \eta_{w_k} \right] + \prod_{i=1}^{r} \eta_{w_i} \tag{62}$$

Proof. With the notation from proposition 121 we have

$$F_w \cap O_{w_i} \subseteq E_{w_i} \cap O_{w_i}$$

and

$$\bigcap_{j=1}^{r_i} F_{w_{i,j}} \subset F_{w_i}$$

as well as

$$
\begin{aligned}
& F_w \cap G_i \\
&\subseteq F_w \cap \bigcap_{k=1}^{i-1} F_{w_k} \cap O_{w_i} \\
&\subseteq \bigcap_{k=1}^{i-1} F_{w_k} \cap (F_w \cap O_{w_i}) \\
&\subseteq \bigcap_{k=1}^{i-1} F_{w_k} \cap (E_{w_i} \cap O_{w_i}) \\
&\subseteq \bigcap_{k=1}^{i-1} F_{w_k} \cap E_{w_i} \cap \left(\bigcap_{j=1}^{r_i} F_{w_{i,j}} \right)^c
\end{aligned}
$$

Now using the above equations and (61) we get

$$
\begin{aligned}
&F_w \\
&= H \uplus \biguplus_{i=1}^{r} \left(F_w \cap H^c \cap \bigcap_{k=i+1}^{r} G_k^c \cap G_i \right) \\
&\subseteq H \uplus \biguplus_{i=1}^{r} (F_w \cap G_i) \\
&\subseteq H \uplus \biguplus_{i=1}^{r} \left[\bigcap_{k=1}^{i-1} F_{w_k} \cap E_{w_i} \cap \left(\bigcap_{j=1}^{r_i} F_{w_{i,j}} \right)^c \right]
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&\mathbf{P}\left(F_w\right) \\
&\leq \mathbf{P}\left(H\right) + \sum_{i=1}^{r} \mathbf{P} \left(\bigcap_{k=1}^{i-1} F_{w_k} \cap E_{w_i} \cap \left(\bigcap_{j=1}^{r_i} F_{w_{i,j}} \right)^c \right) \\
&= \prod_{i=1}^{r} \mathbf{P}\left(F_{w_i}\right) + \sum_{i=1}^{r} \left[\prod_{k=1}^{i-1} \mathbf{P}\left(F_{w_k}\right) \mathbf{P}\left(E_{w_i}\right) \left(1 - \prod_{j=1}^{r_i} \mathbf{P}\left(F_{w_{i,j}}\right) \right) \right] \quad \text{see figure 4} \\
&= \prod_{i=1}^{r} \eta_{w_i} + \sum_{i=1}^{r} \left[\prod_{k=1}^{i-1} \eta_{w_k} q \left(1 - \prod_{j=1}^{r_i} \eta_{w_{i,j}} \right) \right].
\end{aligned}
$$

□

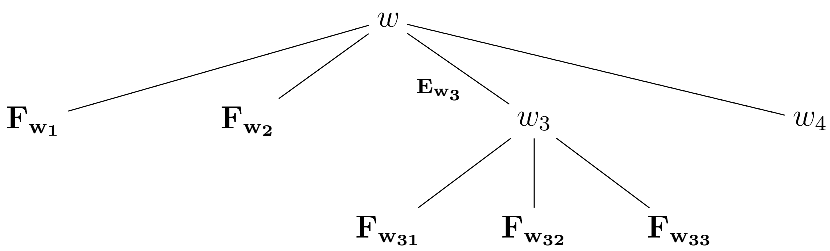

Figure 4: In this example for $r = 4, i = 3$ and $r_3 = 3$ we see that the six events $F_{w_1}, F_{w_2}, E_{w_3}, F_{w_{31}}, F_{w_{32}}$ and $F_{w_{33}}$ have pairwise distance > 1 and are therefore independent of each other.

Lemma 123 (ξ-minoration) *Let T be a finite tree with N full levels rooted at o. For any $\mathcal{P}_p^1(E)$ we have*

$$\forall\, k \in \{1\ldots,N-1\}, \forall\, w \in T_k : \xi_w^N \geq c_k \left(1 - \prod_{i=1}^{r} \eta_{w_i}^N\right) \tag{63}$$

where the $\{c_k\}_{k=1}^N$ are as in the finite case of model 115.

Proof. We prove this by reverse induction on k.
$k = N-1$: Let $w \in T_{N-1}$, then

$$\eta_w^N = \mathbf{P}\left(F_w^N\right) \leq \mathbf{P}\left(E_{w_1}\right) = q = 1 - p = 1 - c_{N-1}$$

$k+1 \to k$: Let $w \in T_k$, then

$$\begin{aligned}
& c_{k+1}\eta_w^N \\
& \leq c_{k+1} q \sum_{k=1}^{r} \left[\left(1 - \prod_{j=1}^{r_k} \eta_{w_{k,j}}^N\right) \prod_{i=1}^{k-1} \eta_{w_i}^N\right] + c_{k+1} \prod_{i=1}^{r} \eta_{w_i}^N && \text{by lemma 122} \\
& \leq q \sum_{k=1}^{r} \left[\left(1 - \eta_{w_i}^N\right) \prod_{i=1}^{k-1} \eta_{w_i}^N\right] + c_{k+1} \prod_{i=1}^{r} \eta_{w_i}^N && \text{induction step} \\
& \leq q \left(1 - \prod_{i=1}^{r} \eta_{w_i}^N\right) + c_{k+1} \prod_{i=1}^{r} \eta_{w_i}^N
\end{aligned}$$

which leads to

$$\begin{aligned}
& \xi_w^N \\
& = 1 - \eta_w^N \\
& \geq 1 - \frac{q}{c_{k+1}} \left(1 - \prod_{i=1}^{r} \eta_{w_i}^N\right) + \prod_{i=1}^{r} \eta_{w_i}^N \\
& = \big(1 - (1 - c_k)\big) \left(1 - \prod_{i=1}^{r} \eta_{w_i}^N\right) && \text{by } c_k = f(c_{k+1}) \\
& = c_k \left(1 - \prod_{i=1}^{r} \eta_{w_i}^N\right)
\end{aligned}$$

□

Lemma 124 (ξ-equality) *Same setting as in lemma 123, but instead of any percolation take finite canonical* 1*-independent percolation with parameter* p*. Then we have the equality*

$$\forall\, k \in \{1\ldots,N-1\}, \forall\, w \in T_k : \xi_w^{N,can(N)} = c_k \left(1 - \prod_{i=1}^{r_i} \eta_{w_i}^{N,can(N)}\right). \tag{64}$$

Proof. Again by reverse induction:
$k = N-1$: Let $w \in T_{N-1}$, then

$$\xi_w^N = \mathbf{P}\left(O_w^N\right) = \mathbf{P}\left(X_w = 1\right) = p = c_{N-1}$$

$k+1 \to k$: Let $w \in T_k$, then

$$\xi_w^N = \mathbf{P}\left(\{X_w = 1\} \cap \left(\bigcap_{i=1}^{r} F_{w_i}^N\right)^c\right) = c_k \left(1 - \prod_{i=1}^{r} \eta_{w_i}^{N,can(N)}\right)$$

by using the structure of the underlying path as detailled in proposition 117 which requires all 1's. □

Corollary 125 (ξ-comparison) *With the same setting as in lemma 123 we have for any* 1*-independent percolation*

$$\forall\, k \in \{1\ldots,N-1\}, \forall\, w \in T_k : \xi_w^N \geq \xi_w^{N,can(N)}. \tag{65}$$

Proof. Again by reverse induction:
$k = N-1$: Let $w \in T_{N-1}$, then using lemmata 123 and 124 we obtain:

$$\xi_w^N \geq c_{N-1}\left(1 - \prod_{i=1}^{r} \eta_{w_i}^N\right) = p\left(1 - \prod_{i=1}^{r} 0\right) = p = \mathbf{P}\left(O_w^{N,can(N)}\right) = \xi_w^{N,can(N)}$$

$k+1 \to k$: Let $w \in T_k$, then

$$\begin{aligned} &\xi_w^N \\ &\geq c_k \left(1 - \prod_{i=1}^{r} \eta_{w_i}^N\right) && \text{by lemma 123} \\ &\geq c_k \left(1 - \prod_{i=1}^{r} \eta_{w_i}^{N,can(N)}\right) && \text{induction step} \\ &= \xi_w^{N,can(N)} && \text{by lemma 124} \end{aligned}$$

□

We are finally at the point where this section can be summed up into

Theorem 126 (ξ-comparison) *Now again on an infinite tree T rooted at o, we have for any $\mathcal{P}_p^1(E)$:*

$$\forall\, w \in V : \xi_w \geq \xi_w^{can(\infty)}. \tag{66}$$

PROOF. Use corollary 125 together with the convergence result for canonical percolation from lemma 119 and the convergence result for general percolation from proposition 75 to extend the statement to the infinite tree. □

As a last note I want to point out another view of the fact that the canonical percolation is the best imaginable:

Proposition 127 *Let T be rooted at o and $w \in V\backslash\{o\}$, w_i one of w's children and $p \in [\frac{3}{4}, 1]$. Then for any $\mathcal{P}_p^1(E)$ we have*

$$\mathbf{P}\left(X_{e(w_i)} = 1 | X_{e(w)} = 1\right) \geq \frac{1-2q}{p}$$

In the case of $\mathcal{P}_p^{1,can}(E)$ the above becomes an equality.

PROOF. For any $\mathcal{P}_p^1(E)$

$$\begin{aligned} \mathbf{P}\left(X_{e(w_i)} = X_{e(w)} = 1\right) &= 1 - \mathbf{P}\left(X_{e(w_i)} = 0 \vee X_{e(w)} = 0\right) \\ &\geq 1 - \left[\mathbf{P}\left(X_{e(w_i)} = 0\right) + \mathbf{P}\left(X_{e(w)} = 0\right)\right] = 1 - 2q \end{aligned}$$

and for $\mathcal{P}_p^{1,can}(E)$ we have for the triple $(Y_w, Y_{w_i}, Y_{w_{i,j}})$ in the underlying $\mathcal{P}_{c(p)}(V)$ only the choice between $(1,1,y), (1,0,0)$ or $(0,0,0)$, hence yielding the equality. □

6.4 $p_{\max}^1(T)$ for $br(T) < 2$

Lemma 128 *If $br(T) < 2$ and $p > g(br(T))$ then any $\mathcal{P}_p^1(E)$ percolation percolates almost surely.*

PROOF. As $br(T) < 2$ we have $\frac{3}{4} < g(br(T)) < p$ and by proposition 113 we get $c(p) > \frac{1}{br(T)}$. From here on there are 2 ways to finish the proof, both by showing that on a rooted version of T $\mathcal{P}_p^{1,can}(E)$ has downrays from the root o with positive probability. The final step common to both versions is then the application of theorem 126 to establish percolation for all other $\mathcal{P}_p(E) \in \mathcal{C}_p^1(E)$ on said rooted version of T and finally show almost sure percolation on the unrooted version of T using theorem 78.

Bollobas & Ballister: Root T at o. For $\lambda \in]\frac{1}{c(p)}, br(T)[$ take a λ-flow f from o on T. Set $\varepsilon = c(p) - \frac{1}{\lambda} \in]0,1[$ for an appropriate choice of λ. We now claim that for any $\mathcal{P}_p^1(E)$ we have

$$\forall\, w \in V : \xi_w \geq \varepsilon \lambda^{l_v} f(w).$$

We are first showing this fact for $T|_N$, where $N \in \mathbb{N}$ and then again lifting it to the infinite tree via lemma 119, thus we start with showing that

$$\forall\, N \in \mathbb{N} : \forall\, w \in V(T|_N) : \xi_w^N \geq \varepsilon \lambda^{l_w} f(w).$$

This we prove by reverse induction:
$k = N : \forall\, w \in T|_N^N : \xi_w^N = 1 \geq \varepsilon \lambda^N f(w)$
$k+1 \to k$: Let $w \in T|_N^k$, then

$$\begin{aligned}
&\xi_v^N \\
&= 1 - \eta_v^N \\
&\geq c_k \left(1 - \prod_{i=1}^{r} \eta_{v_i}^N \right) && \text{by lemma 123} \\
&\geq c(p) \left(1 - \prod_{i=1}^{r} \eta_{v_i}^N \right) && \text{by 114: } c_k > c(p) \\
&\geq c(p) \left[1 - \prod_{i=1}^{r} \left(1 - \varepsilon \lambda^{k+1} f(v_i) \right) \right] && \text{by induction hypothesis} \\
&\geq c(p) \left(1 - \prod_{i=1}^{r} e^{-\varepsilon \lambda^{k+1} f(v_i)} \right) && \text{as } \forall\, y \in \mathbb{R}_+ : e^{-y} \geq 1 - y \\
&= c(p) \left(1 - e^{-\varepsilon \lambda^{k+1} \sum_{i=1}^{r} f(v_i)} \right) \\
&= c(p) \left(1 - e^{-\varepsilon \lambda^{k+1} f(v)} \right) && \text{by the definition of flow 27} \\
&\geq c(p) \frac{\varepsilon \lambda^{k+1} f(v)}{1 + \varepsilon \lambda^{k+1} f(v)} && \text{as } \forall\, y \in \mathbb{R}_+ : 1 - e^{-y} \geq \frac{y}{1+y} \\
&= \varepsilon \lambda^k f(v) \frac{c(p)\lambda}{1 + \varepsilon \lambda \lambda^k f(v)} \\
&\geq \varepsilon \lambda^k f(v) \frac{c(p)\lambda}{1 + \varepsilon \lambda} && f\lambda\text{-flow } \Rightarrow f(v)\lambda^k \leq 1 \\
&= \varepsilon \lambda^k f(v) && \text{by definition of } \varepsilon
\end{aligned}$$

We now use theorem 75 to extend this inequality to the infinite tree, thus we have $\forall\, w \in V : \xi_w \geq \varepsilon$ upon taking $f(v) = \lambda^{-l_v}$. Finally apply theorem 78 to show percolation on the unrooted tree.

Pierre Mathieu: As $c(p) > \frac{1}{br(T)}$ we have percolation for $\mathcal{P}_{c(p)}(V)$ upon application of theorem 102. Rooting T at o and using theorem 78 we now find that we have a downray from o with positive probabiliy $\xi_o > 0$. This downray translates itself into a downray for $\mathcal{P}_p^{1,can}(E)$, thus we have

$$0 < \xi_o^{can(\infty,1)} \leq \xi_o^{can(\infty)}.$$

$\square$

Lemma 129 *Let T have $br(T) < 2$ and $p \in]\frac{3}{4}, g(br(T))[$. Then $\mathcal{P}_p^{1,can}(E)$ doesn't percolate.*

Proof. By proposition 113 we get $c(p) \in]\frac{1}{2}, \frac{1}{br(T)}[$. By proposition 120 we only have to care about underlying rays of 1's. These exist only if $\mathcal{P}_{c(p)}(V)$ percolates, which it doesn't (use theorem 102 and $c(p) < \frac{1}{br(T)}$). $\square$

Theorem 130 *Let T be a locally finite, leafless tree with $br(T) < 2$, then*

$$p_{\max}^1(T) = g(br(T)) = 1 - \frac{br(T) - 1}{br(T)^2}. \tag{67}$$

Proof. Combine the results of lemmata 128 and 129. $\square$

Corollary 131 *If $br(T) \geq 2$, then $\frac{3}{4} \leq p_{\max}^1(T)$.*

Proof. Let $\frac{3}{4} < p$. As the branching number is a monotone growing property of trees we can choose a substree T' of T with $br(T') < 2$ such that $g(br(T')) < p$ (works for all $\frac{3}{4} < p$ by proposition 112). By theorem 130 we percolate on T', hence we percolate on T (as we can fill up $T \backslash T'$ with independent Bernoulli random variables and percolation is a monotone growing property of graphs, too). $\square$

6.5 The impossible model

Model 132 *We call the* **impossible model of 1-independent percolation**, $\mathcal{P}^{1,imp}(E)$*, the following model: starting from the root o, assign marks to the nodes via the function*

$$m: \qquad V \to \{0, 1, \star\} \qquad v \mapsto \begin{cases} 0 & l_v \equiv 2, 3 \mod 6 \\ 1 & l_v \equiv 5, 6 \mod 6 \\ \star & l_v \equiv 1 \mod 3 \end{cases} \tag{68a}$$

Now assign independent random variables $\{X_v\}_{v\in V}$ of the following form to the nodes

$$m_v = 0 \Leftrightarrow X_v \sim Bernoulli\left(\frac{1}{\sqrt{2}}\right)$$
$$m_v = 1 \Leftrightarrow X_v \sim Bernoulli\left(1-\frac{1}{\sqrt{2}}\right)$$
$$m_v = \star \Leftrightarrow X_v \sim Bernoulli\left(\frac{1}{2}\right)$$

and define the variables for $\mathcal{P}^{1,imp}(E)$, $\{Y_e\}_{e\in E}$, by

$$\{Y_e = 1\} = \{X_{v(e)} = X_{p(v(e))} \wedge \neg(X_{v(e)} = X_{p(v(e))} = m_{v(e)} = m_{p(v(e))})\}. \tag{68b}$$

For a visualization of this model see figure 5 on page 54. This model not only works on trees, but can be applied to any graph, therefore the lower bound on $p^1_{\max}(T)$ it delivers (see proposition 133) is universal. Shorter models can be found on trees (see notes after proposition 138).

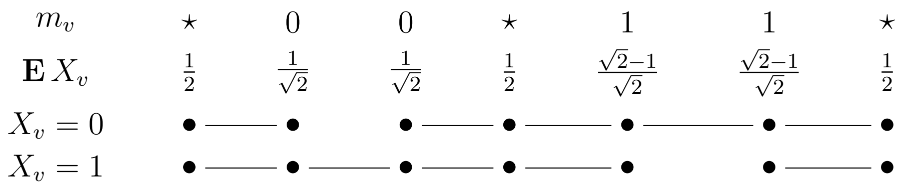

Figure 5: Visualization of the impossible percolation along a ray section

Proposition 133 *$\mathcal{P}^{1,imp}(E)$ as defined in 132 has parameter $\frac{1}{2}$.*

Proof. As the definition in (68b) is symmetric in the two endnodes v, w of an edge e, we only have to regard the following cases (ordered by (m_v, m_w)):

$$\mathbf{P}(Y_e = 1)$$
$$= \begin{cases} \mathbf{P}(X_v = X_w = 1) = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} & (0,0) \\ \mathbf{P}(X_v = X_w = 0) = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} & (1,1) \\ \mathbf{P}(X_v = X_w = 0) + \mathbf{P}(X_v = X_w = 1) = \frac{1}{2}\left(\frac{1}{\sqrt{2}} + 1 - \frac{1}{\sqrt{2}}\right) = \frac{1}{2} & (0,\star) \\ \mathbf{P}(X_v = X_w = 0) + \mathbf{P}(X_v = X_w = 1) = \frac{1}{2}\left(\frac{1}{1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}}\right) = \frac{1}{2} & (1,\star) \end{cases}$$

□

Lemma 134 $\mathcal{P}^{1,imp}(E)$ *as defined in 132 doesn't percolate.*

PROOF. Denote by $C = (V_C, E_C)$ any connected component of $\mathcal{P}^{1,imp}(E)$. It's plain to see that all $\{X_v\}_{v \in V_C}$ must realize in the same value. Assume that this value is 0, hence C can't contain any of the edges whose endpoints are both marked with 0. These edges separate T into finite rings around o and C is contained within one of these rings, therefore C is finite. The same reasoning applies for the value 1, hence $\mathcal{P}^{1,imp}(E)$ doesn't percolate as all of its components are deterministically finite. □

6.6 $p^1_{\min}(T)$

Model 135 *Take a* $\mathcal{P}_{\sqrt{p}}(V)$ $\{X_v\}_{v \in V}$ *and construct from it the* **minimal model of 1-independent bond percolation with parameter** p*, abbreviated* $\mathcal{P}^{1,\min}_p(E)$*,* $\{Y_e\}_{e \in E}$ *by*

$$\{Y_e = Y_{e(v)} \text{ open}\} = \{X_v = 1 \wedge X_{p(v)} = 1\}. \tag{69}$$

Theorem 136 *We have*

$$p^1_{\min}(T) = \frac{1}{br(T)^2}. \tag{70}$$

PROOF.

"$\leq$": Suppose $p > \frac{1}{br(T)^2}$. Then the underlying $\mathcal{P}_{\sqrt{p}}(V)$ $\{X_v\}_{v \in V}$ of $\mathcal{P}^{1,\min}_p(E)$ $\{Y_e\}_{e \in E}$ percolates by theorem 102 as $\sqrt{p} > \frac{1}{br(T)}$. As $\{X_v\}_{v \in V}$ percolates, it has a node w of finite rank from with a ray of 1's from $w \to \infty$ starts. This translates itself by (69) to a ray of open edges in $\{Y_e\}_{e \in E}$, hence it percolates, too (using two times theorem 78).
"$\geq$": For any $\mathcal{P}^1_p(E)$ the 1^{st} moment method (see 81) results in

$$\begin{aligned}
&\mathbf{P}\left(o \leftrightarrow \infty\right) \\
&\leq \inf_{\Pi \in \Pi(o)} \sum_{e \in \Pi} \mathbf{P}\left(e \in C(o)\right) \\
&\leq \inf_{\Pi \in \Pi(o)} \sum_{e \in \Pi} (\sqrt{p})^{l_{v(e)}} && \text{by (12) in proposition 64} \\
&\leq \inf_{\Pi \in \Pi(o)} \sum_{e \in \Pi} \left(\frac{1}{\sqrt{p}}\right)^{-l_{v(e)}}
\end{aligned}$$

which implies that we have

$$\begin{aligned}
&\mathcal{P}_p^1(E) \text{ percolates} \\
\Leftrightarrow\ & \mathbf{P}\left(o \leftrightarrow \infty\right) > 0 && \text{by theorem 78} \\
\Rightarrow\ & \inf_{\Pi \in \Pi(o)} \sum_{e \in \Pi} \left(\frac{1}{\sqrt{p}}\right)^{-l_{v(e)}} > 0 \\
\Rightarrow\ & \frac{1}{\sqrt{p}} \leq br(T) && \text{by definition 40} \\
\Leftrightarrow\ & p \geq \frac{1}{br(T)^2}.
\end{aligned}$$

□

6.7 Bernoulli models

All the models of 1-independent percolation introduced so far (see models 115, 132 and 135) have had the same structure: Mark the nodes repeatingly with the same labels from a finite set (may have cardinality 1), then assign Bernoulli random variables with parameters depending on the marks to them and finally construct the Bernoulli random variables with parameter p for the edges from the Bernoulli random variable of their respective endpoints. From now on I will call these models **Bernoulli models**, referring to their construction.

Using the enumeration from table 1, we could write model 115 as a repetition of #14 (only the infinite case), model 132 as a repetition of the sequence (#6, #5, #6, #6, #2, #6) going out from o and model 135 as #5.

Next we want to classify the behaviour of Bernoulli models and detail some of the restrictions it imposes. Let $e \in E$ with endpoints $v, w \in V$ and assign Bernoulli random variables with parameters p_v and p_w to v and w: $X_v \sim$ Bernoulli(p_v) and $X_w \sim$ Bernoulli(p_w). To construct now a $Y_e \sim$ Bernoulli(p) from X_v and X_w we have only a σ-algebra with 4 elements at our disposition (see table 1 on page 57).

Definition 137 *A percolation $\mathcal{P}$ on a tree T with root o is called* **impossible** *iff along any ray $R \in \Upsilon(o)$ we have a deterministic finite section S_R such that $\mathbf{P}\left(S_R \text{ open}\right) = 0$.*

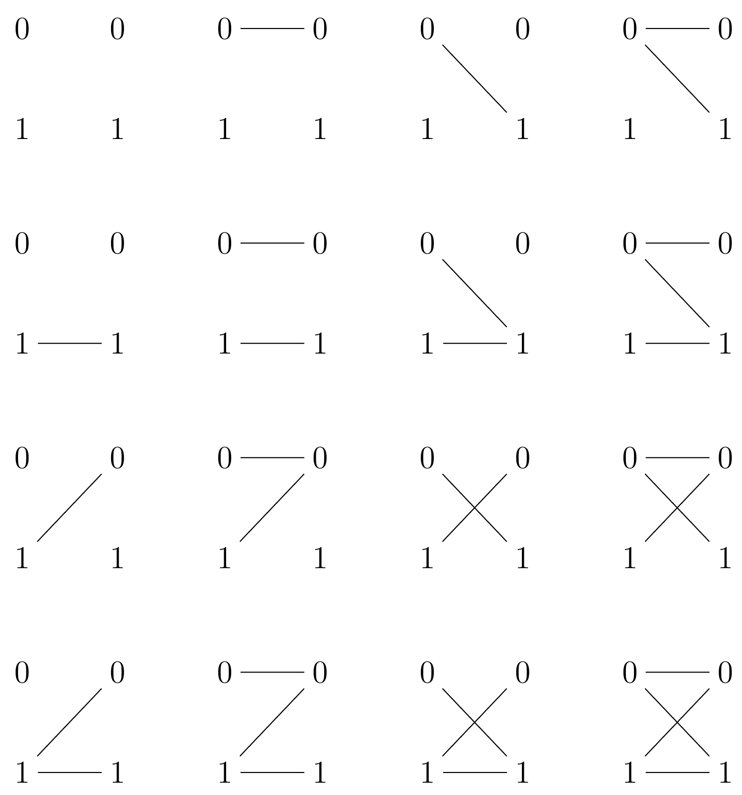

Table 1: These are the 16 **events** in the σ-algebra, enumerated from event #1 to event #16 in reading order, of a Bernoulli random variable generated from two independent Bernoulli random variables

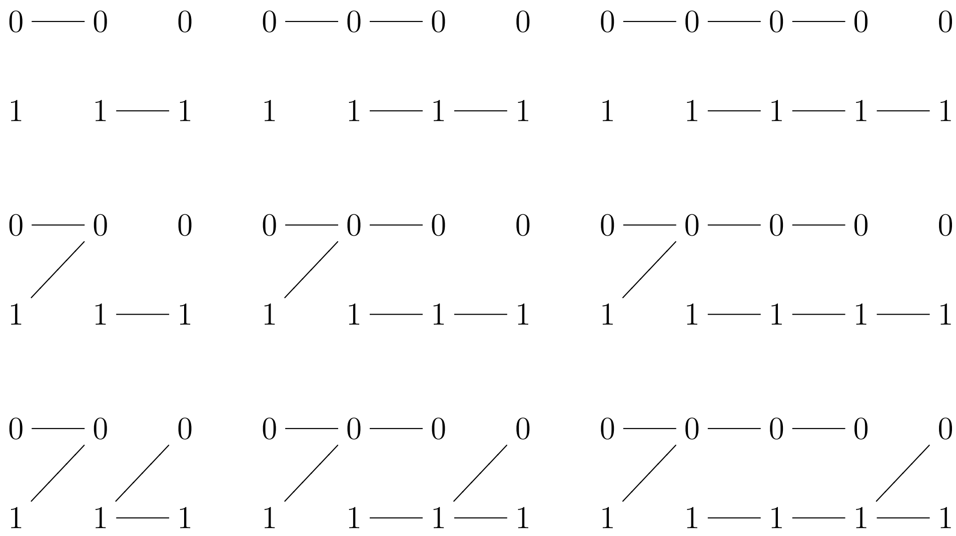

Table 2: In each line, we have one of the three possible cases modulo symmetries, to have a cut, with examples of $\Omega^1_{S_R}$ and $\Omega^2_{S_R}$ running parallel for $0, 1$ or 2 edges from left-to-right.

Proposition 138 *Any impossible $\mathcal{P}^1_p(E) \in \mathcal{C}^1_p(E)$ based on Bernoulli models has parameter $p \le \frac{1}{2}$.*

Proof. Root T at o and regard $R \in \Upsilon(o)$. We need a finite section S_R of R such that $\mathbf{P}(S_R \text{ open}) = 0$. From the construction of Bernoulli models it follows that S_R has to contain at least two disconnected sets $\Omega^1_{S_R}$ and $\Omega^2_{S_R}$ of events in the underlying σ-algebra. The cut between these has the following structure: the set $\Omega^1_{S_R}$ arrives from the left, then $\Omega^2_{S_R}$ is starting, they may run for some time in parallel and then $\Omega^1_{S_R}$ ends and only $\Omega^2_{S_R}$ rests (see table 2 on page 58 for examples). We demonstrate that for all possible cuts the restrictions force us to choose $p \le \frac{1}{2}$. The parameters for the Bernoulli random variables on the nodes will be denoted by $c_0, c_1, c_2, \ldots$, starting from the left, and the parameter of the bond variables will be p.

Cases with parallel run of length 0: left column in table 2
Upper case: $p = (1-c_0)(1-c_1) \wedge p = c_1 c_2 \Rightarrow p \le 1 - c_1 \wedge p \le c_1 \Rightarrow p \le \frac{1}{2}$.
Middle case: $p = 1 - c_1 \wedge p = c_1 c_2 \Rightarrow p = 1 - c_1 \wedge p \le c_1 \Rightarrow p = c_1 = \frac{1}{2}$.
Lower case: $p = 1 - c_1 \wedge p = c_1 \Rightarrow p = c_1 = \frac{1}{2}$.

Cases with parallel run of length 1: center column in table 2
Upper case: $p = (1-c_0)(1-c_1) \wedge p = (1-c_1)(1-c_2) + c_1 c_2 \wedge p = c_2 c_3$.
It follows that $c_1 = \frac{1-c_0-p}{1-c_0}, c_2 = \frac{pc_0}{2p+c_0-1}$ and $c_3 = \frac{2p+c_0-1}{c_0}$ which results in

$1 \geq c_3 = \frac{2p+c_0-1}{c_0} \Rightarrow p \leq \frac{1}{2}$.
Middle case: $p = 1 - c_1 \wedge p = (1-c_1)(1-c_2) + c_1c_2 \wedge p = c_2c_3$.
Hence $p = p(1-c_2) + (1-p)c_2 \Rightarrow p = \frac{1}{2}$.
Lower case: $p = 1 - c_1 \wedge p = (1-c_1)(1-c_2) + c_1c_2 \wedge p = c_2$.
Therefore we have $p = p(1-p) + (1-p)p \Rightarrow p = \frac{1}{2}$.

Cases with parallel run of length 2 or longer: Examples with a parallel run of length 2 can be seen in the right column of table 2. Note that in the case of longer parallel runs we have $c_1 = c_3 = c_5 = \ldots$ and $c_2 = c_4 = c_6 = \ldots$ by symmetry in the equations for events #6 and #11. Hence treatment of all those cases is already included in the treatment of the length 0 and 1 cases. □

Corollary 139 *Any percolation based on a Bernoulli model with parameter $p > \frac{1}{2}$ on any tree has downpaths of all lengths open with positive probability.*

Proof. Negation of proposition 138. □

The shortest possible Bernoulli model on a tree with $p = \frac{1}{2}$ is given by the repetition of the following sequence of events: #4, #10, #13, #7. If one wants a percolation such that $\forall\, n \in \mathbb{N} : \mathbf{P}\left(Y_n = 1 | Y_{n-1} = \ldots = Y_1 = 1\right) = \mathbf{P}\left(Y_n = 1 | Y_{n-1}\right)$ then the only candidates are percolations based on events #12 or #14.

6.8 A related problem

A related problem to 1-independent percolation is the following: denoting n linear 1-independent Bernoulli random variables with parameter p by $\{Y_i\}_{i=1}^n$, we may ask for the value of

$$p_n = \max\left\{p : \forall\, \{Y_i\}_{i=1}^n \text{as above } \mathbf{P}\left(Y_1 = \ldots = Y_n = 1\right) = 0\right\}?$$

We can give some bounds for p_n. If $p > \frac{3}{4}$, then

$$\begin{aligned} &\mathbf{P}_p(Y_1 = \ldots = Y_n = 1) \\ &\geq \mathbf{P}_p^{can(N)}(Y_1 = \ldots = Y_n = 1) && \text{by lemmata 123 and 124} \\ &= \prod_{i=1}^n c_i \geq \prod_{i=1}^n c(p) && \text{see model 115} \\ &= c(p)^n > 0 && \text{by proposition 114,} \end{aligned}$$

hence $\forall\, n \in \mathbb{N} : p_n \leq \frac{3}{4}$. On the other hand, we have $p_2 = \frac{1}{2}$, $p_3 = \frac{\sqrt{5}-1}{2}$, $p_4 = \frac{2}{3}$ and $p_5 \sim 0.73205$ (approximated by a linear program) and in general

$\forall\, n \in \mathbb{N} : p_n \leq p_{n+1}$, but at the moment it is not clear whether $p_n \xrightarrow[n\to\infty]{} \frac{3}{4}$. As this problem is equivalent to finding a 1-independent percolation with parameter $\frac{3}{4}$ on a ray which never percolates, it embodies the essential aspects of the problem of determining if $p^1_{\max}(T)$ is indeed $\frac{3}{4}$.

7 K-independent percolation on trees

Here we present the theorems about k-independent percolation for $k > 1$. Thoughts and musings are treated in section 8.

7.1 $p^k_{\min}(T)$ for $k \geq 2$

As usual, let $T = (V, E)$ be a leafless, locally finite tree with finite branching number and if necessary rooted at o. The results in this section subsume the results of section 6.6 and extend them to general k.

Model 140 *Add a path of length k at o to T and call the resulting tree $T' = (V', E')$. Proposition 44 asserts $br(T') = br(T)$. Root T' at the leaf-node o' of the added path. Take a $\mathcal{P}_{\sqrt[k+1]{p}}(E')$ $\{X_e\}_{e \in E'}$ for $\sqrt[k+1]{p} > \frac{1}{br(T)}$ and construct the* **minimal model of k-independent bond percolation with parameter** *p, abbreviated $\mathcal{P}^{k,\min}_p(E)$, $\{Y_e\}_{e \in E}$ from it by setting*

$$\{Y_e = 1\} = \{X_e = X_{f_1} = \ldots = X_{f_k} = 1\} \Rightarrow \mathbf{P}\left(Y_e = 1\right) = p, \tag{71}$$

where $f_1, \ldots, f_k$ are the k next edges on the path from $p(v(e))$ to o' in T'.

Lemma 141

$$\forall\, k \in \mathbb{N}: \quad p^k_{\min}(T) \leq \frac{1}{br(T)^{k+1}}.$$

PROOF. For the $\mathcal{P}^{k,\min}_p(E)$ as constructed in 140 we have

$$\begin{aligned} &\{X_e\}_{e \in E'} \text{ percolates} && \text{by theorem 102} \\ \Leftrightarrow\; &\mathbf{P}\left(o' \leftrightarrow \infty\right) > 0 \text{ for } \mathcal{P}_{\sqrt[k+1]{p}}(E') && \text{by theorem 78} \\ \Rightarrow\; &\mathbf{P}\left(o \leftrightarrow \infty\right) > 0 \text{ for } \mathcal{P}^k_p(E) && \text{by the construction} \\ \Leftrightarrow\; &\{Y_e\}_{e \in E} \text{ percolates} && \text{by theorem 78.} \end{aligned}$$

We conclude as we find such a percolating $\mathcal{P}^k_p(E)$ for all $p > \frac{1}{br(T)^{k+1}}$. $\square$

Lemma 142

$$\forall\, k \in \mathbb{N}: \quad p^k_{\min}(T) \geq \frac{1}{br(T)^{k+1}}.$$

Proof. For any $\mathcal{P}_p^k(E)$ the 1^{st} moment method (see 81) results in

$$
\begin{aligned}
&\mathbf{P}\left(o \leftrightarrow \infty\right) \\
\leq &\inf_{\Pi \in \Pi(o)} \sum_{e \in \Pi} \mathbf{P}\left(e \in C(o)\right) \\
\leq &\inf_{\Pi \in \Pi(o)} \sum_{e \in \Pi} p^{\left\lceil \frac{l_{v(e)}}{k+1} \right\rceil} && \text{by (12) in proposition 64} \\
\leq &\inf_{\Pi \in \Pi(o)} \sum_{e \in \Pi} \left(p^{-\frac{1}{k+1}}\right)^{-(k+1)\left\lceil \frac{l_{v(e)}}{k+1} \right\rceil} \\
\leq &\inf_{\Pi \in \Pi(o)} \sum_{e \in \Pi} \left(p^{-\frac{1}{k+1}}\right)^{-l_{v(e)}} && \text{as } (k+1)\left\lceil \frac{l_{v(e)}}{k+1} \right\rceil > l_{v(e)}
\end{aligned}
$$

which implies that we have

$$
\begin{aligned}
&\mathcal{P}_p^k(E) \text{ percolates} \\
\Leftrightarrow &\,\mathbf{P}\left(o \leftrightarrow \infty\right) > 0 && \text{by theorem 78} \\
\Rightarrow &\inf_{\Pi \in \Pi(o)} \sum_{e \in \Pi} \left(p^{-\frac{1}{k+1}}\right)^{-l_{v(e)}} > 0 \\
\Rightarrow &\, p^{-\frac{1}{k+1}} \leq br(T) && \text{by definition 40} \\
\Leftrightarrow &\, p \geq \frac{1}{br(T)^{k+1}}.
\end{aligned}
$$

□

Theorem 143

$$\forall\, k \in \mathbb{N}: \quad p_{\min}^k(T) = \frac{1}{br(T)^{k+1}}.$$

Proof. Combine lemmata 141 and 142. □

8 Summary and outlook

8.1 Summary of results

The principal results are summarized in table 3 and plotted in figure 6. There are a few observations one can make from looking at all those results together.

First, one clearly sees the ordering of critical values for the different percolation classes as stated in lemma 80.

Then one can see the uniformity of the results for $p_c(T) = p^0_{\min}(T)$, $p^1_{\min}(T)$ and $p^2_{\min}(T)$ which is due to the the common use of the 1^{st} moment method in the proof of all of these (see proposition 98, theorem 136 and lemma 142 for the respective uses for $k = 0, 1, 2$).

Finally, one can see that the same can't be said for $p_c(T) = p^0_{\max}(T)$ and $p^1_{\max}(T)$: the exact value of $p^1_{\max}(T)$ isn't even known for $2 < br(T)$ (indicated by an "?" in figure 6), except for the fact that it lies closely below $\frac{3}{4}$ (read also the discussion in section 6.8).

$p^2_{\min}(T) = \dfrac{1}{br(T)^3}$	theorem 143 for $k = 2$ on page 62
$p^1_{\min}(T) = \dfrac{1}{br(T)^2}$	by theorem 136 on page 55
$p_c(T) = \dfrac{1}{br(T)}$	by theorem 102 on page 34
$p^1_{\max}(T) = 1 - \frac{br(T)-1}{br(T)^2}$ for $br(T) \in [1, 2]$	by theorem 130
$p^1_{\max}(T) \in [\frac{1}{2}, \frac{3}{4}]$ for $2 < br(T)$	corollary 131 and proposition 133

Table 3: Summary of Results

8.2 Open questions

There are two main questions remaining open: The value of $p^2_{\max}(T)$ for $2 < br(T)$ (indicated by "?" in figure 6 on page 64) and knowledge about $p^k_{\max}(T)$ for higher k.

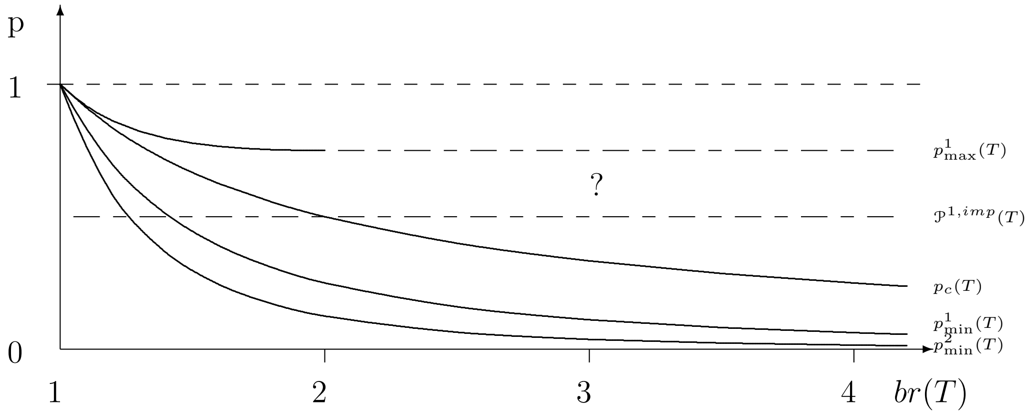

Figure 6: Plot of the critical values: see their definitions in table 3

At the core of the first problem is the inadequacy of the subset of the Bernoulli models from section 6.7 to capture all critical aspects of 1-independent bond percolation. The tighest bounds they yield imply that $p^1_{\max}(T) \in [\frac{1}{2}, \frac{3}{4}]$ by using the impossible model 132 on page 53. The best current guess (see section 6.8) would be that indeed $p^1_{\max}(T) = \frac{3}{4}$, but no proof is known to me yet.

Two possible directions open from there: the first would be to look at general models of 1-independent Bernoulli variables and construct from them stricter lower bounds. This would imply investigating their σ-algebras and trying to build models directly on them. The second direction would be to find the subclass of 1-independent percolations for which once can show that they tend asymptotically to $\frac{1}{2}$ (via exponential intersection tails and a suitable percolation ray kernel).

As for the second question, the most direct approach would be to try to mimic the behaviour of the canonical model 115 detailled in proposition 127 by searching for 2-independent models which minimize $\mathbf{P}(Y_{i+3} = 1 | Y_{i+2} = Y_{i+1} = 1)$ or $\mathbf{P}(Y_{i+3} = Y_{i+2} = 1 | Y_{i+1} = 1)$ for any three Bernoulli random variables $Y_{i+3}, Y_{i+2}, Y_{i+1}$ along a ray. Here again the most simple class of models would be the one built from independent Bernoulli random variables on edges: $\forall\, e \in E : Y_e = F_e(\{X_f\}_{f \backsim e})$.

References

[1] N. Alon, H. J. Spencer, and P. Erdös. *The Probabilistic Method*. John Wiley & Sons, 1992.

[2] B. Bollobás and N. P. Ballister. notes from their presentation taken by Mathieu, Pierre, 2005.

[3] B. Bollobás and O. Riordan. *Percolation*. Cambrigde University Press, 2006.

[4] P. Erdös and A. Rènyi. On the evolution of random graphs. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 5:17–61, 1960.

[5] L. Ford, Jr. and D. Fulkerson. *Flows in Networks*. Princeton Univ. Press, 1962.

[6] Fürstenberg. Intersections of cantor sets and transversality of semigroups. *Problems in Analysis*, page 41, 1970.

[7] V. Giradin and N. Limnios. *Probabilités*. Librairie Vuibert, 2001.

[8] G. Grimmet. *Percolation*. A Series of Comprehensive Studies in Mathematics. Springer, second edition, 1991.

[9] R. Lyons and Y. Peres. *Probability on trees and networks*. published on the web, 2005.

[10] Y. Peres. Course Notes: Probability on Trees and Networks (Berkeley), 2004.

[11] Y. Peres. Probability on Trees: An Introductory Climb (Saint Flour Summer School). pages 193–280. Springer, 2004.

[12] A. N. Shiryaev and R. P. Boas. *Probability*. Graduate texts in mathematics. Springer, second edition, 1991.

[13] C. Temmel. La composante geante d'un graphe aleatoire, 2007. TER dans le M1.